

REGULAR DECOMPOSITIONS FOR $H(\text{div})$ SPACES[†]

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ABSTRACT. We study regular decompositions for $H(\text{div})$ spaces. In particular, we show that such regular decompositions are closely related to a previously studied “inf-sup” condition for parameter-dependent Stokes problems, for which we provide an alternative, more direct, proof.

1. THE MAIN RESULT

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ be a polygonal/polyhedral Lipschitz domain, on which we define the standard Sobolev spaces $L_2(\Omega)$, $H_0^1(\Omega)$, $\mathbf{L}^2(\Omega) \equiv (L_2(\Omega))^d$, and $\mathbf{H}_0^1(\Omega) \equiv (H_0^1(\Omega))^d$ with norms $\|\cdot\|_0$ and $\|\cdot\|_1$, as well as the Hilbert space $\mathbf{H}_0(\Omega, \text{div})$ consisting of $\mathbf{L}^2(\Omega)$ vector-functions \mathbf{v} that have divergence, $\text{div } \mathbf{v}$, in $L_2(\Omega)$ and vanishing normal trace $\mathbf{v} \cdot \mathbf{n} = 0$ on the boundary $\partial\Omega$. For $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$, $\mathbf{u} \in \mathbf{H}_0(\Omega, \text{div})$ and a fixed $\tau > 0$, we also introduce the parameter-dependent symmetric quadratic forms

$$(1.1) \quad \mathcal{L}_\tau(\mathbf{z}, \mathbf{z}) \equiv \|\mathbf{z}\|_0^2 + \tau \|\text{grad } \mathbf{z}\|_0^2 \quad \text{and} \quad \mathcal{L}_\tau^{\text{div}}(\mathbf{u}, \mathbf{u}) \equiv \|\mathbf{u}\|_0^2 + \tau \|\text{div } \mathbf{u}\|_0^2.$$

We denote the corresponding norms with $\|\cdot\|_{\mathcal{L}_\tau}$ and $\|\cdot\|_{\mathcal{L}_\tau^{\text{div}}}$.

The goal of this paper is to prove the following regular decomposition result which is the key ingredient in the construction of the auxiliary space HX-preconditioners for $H(\text{div})$ problems in [HX07]. Similar regular decompositions were needed in the construction of auxiliary space HX-preconditioners for $H(\text{curl})$ problems (cf., [HX07] and [KV09]).

Theorem 1.1. *Given $\mathbf{u} \in \mathbf{H}_0(\Omega, \text{div})$ and a fixed $\tau > 0$, there is a $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$ such that $\text{div } \mathbf{z} = \text{div } \mathbf{u}$ and*

$$(1.2) \quad \|\mathbf{z}\|_{\mathcal{L}_\tau} \leq \frac{1}{c_0} \|\mathbf{u}\|_{\mathcal{L}_\tau^{\text{div}}}$$

The above \mathbf{z} depends on τ , whereas the constant $c_0 > 0$ is independent of τ .

Our construction of \mathbf{z} is based on a certain “inf-sup” stability result associated with the following \mathcal{L}_τ -based parameter-dependent Stokes problem:

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[†]*Dedicated to Sergey Nepomnyaschikh—a pioneer in domain decomposition methods.*

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Given $\mathbf{u} \in \mathbf{H}_0(\Omega, \operatorname{div})$, find $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$ and $\bar{p} \in L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_{\Omega} q \, d\mathbf{x} = 0\}$ such that

$$(1.3) \quad \begin{aligned} (\mathbf{z}, \boldsymbol{\theta}) + \tau (\nabla \mathbf{z}, \nabla \boldsymbol{\theta}) + (\bar{p}, \operatorname{div} \boldsymbol{\theta}) &= 0, & \forall \boldsymbol{\theta} \in \mathbf{H}_0^1(\Omega), \\ (\operatorname{div} \mathbf{z}, q) &= (\operatorname{div} \mathbf{u}, q), & \forall q \in L_0^2(\Omega). \end{aligned}$$

More specifically, if we define the (scalar) Laplace operator Δ_N with homogeneous Neumann boundary conditions (see Section 2 and 3 for details), then the following characterization holds.

Theorem 1.2. *The validity of the “inf-sup” estimate*

$$(1.4) \quad c_0 \left((\tau I - \Delta_N^{-1})^{-1} \bar{p}, \bar{p} \right)^{\frac{1}{2}} \leq \sup_{\boldsymbol{\theta} \in \mathbf{H}_0^1(\Omega)} \frac{(\bar{p}, \operatorname{div} \boldsymbol{\theta})}{\|\boldsymbol{\theta}\|_{\mathcal{L}_\tau}},$$

with a constant $c_0 > 0$ uniform with respect to the parameter τ , is equivalent with the statement of Theorem 1.1, i.e. that for any given $\tau > 0$ and $\mathbf{u} \in \mathbf{H}_0(\Omega, \operatorname{div})$, there is a $\mathbf{z} = \mathbf{z}_\tau \in \mathbf{H}_0^1(\Omega)$ such that $\operatorname{div} \mathbf{z} = \operatorname{div} \mathbf{u}$ and (1.2) holds.

The inf-sup condition (1.4) has been investigated in several papers previously, most notably for convex domains Ω , see [OPR06], [MW04]–[MW06], and [MW11]. The case of general Lipschitz domains has been dealt with recently in [MSW11] based on explicit representation of right inverse of the divergence operator provided by the so-called Bogovskii integral operator. The purpose of the present note is to give a somewhat more direct proof of the same “inf-sup” estimate relating it, as in Theorem 1.2, to the regular decomposition of $\mathbf{H}_0(\Omega, \operatorname{div})$ defined in Theorem 1.1.

The remainder of the present paper is organized as follows. In Section 2, we provide a main stability estimate for $H(\operatorname{div})$ functions with piecewise constant divergence. Section 3 contains the proof of Theorem 1.2. A proof of our main result, Theorem 1.1, for domain that is union of two domains for which the result is valid, is given in Section 4 under assumption (A) which is verified in the following Section 5. The paper concludes with Section 6 where we consider the somewhat simpler case of “large” parameter τ , i.e., τ being much bigger than the diameter of the domain Ω .

2. PRELIMINARIES

The Laplace operator with homogeneous Neumann boundary conditions, Δ_N , is invertible for any right-hand side $f \in L_0^2(\Omega)$ and satisfies $\psi = (-\Delta_N)^{-1} f \in H^1(\Omega) \cap L_0^2(\Omega)$. Moreover, there is a $\delta \in (0, \frac{1}{2}]$ such that we have the regularity estimate

$$\|\psi\|_{\frac{3}{2}+\delta} \leq C \|f\|_{-\frac{1}{2}+\delta}.$$

This result is proved in [Do88], Corollary 23.5, and stated as Lemma A.53 in [TW05].

Let \mathcal{T}_H be a quasiuniform triangulation of Ω with triangles or tetrahedrons of mesh size H . We associate with \mathcal{T}_H the well-known lowest-order Raviart–Thomas space \mathbf{R}_H (with vanishing normal traces on $\partial\Omega$), i.e., $\mathbf{R}_H \subset \mathbf{H}_0(\Omega, \operatorname{div})$. Let $\boldsymbol{\Pi}_H$ be the natural Raviart–Thomas interpolation operator, which is well-defined for sufficiently smooth functions. In particular, $\boldsymbol{\Pi}_H \mathbf{g}$ is well-defined for $\mathbf{g} = \nabla \psi \in \mathbf{H}^{\frac{1}{2}+\delta}(\Omega)$, the

fractional Sobolev space obtained by interpolation between $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^1(\Omega)$, and the following approximation result holds (cf. Theorem 5.25 in [Mo03])

$$(2.1) \quad \|\mathbf{g} - \mathbf{\Pi}_H \mathbf{g}\|_0 \leq CH^{\frac{1}{2}+\delta} \|\mathbf{g}\|_{\frac{1}{2}+\delta}.$$

Let \bar{V}_H be the space of piecewise constant functions associated with \mathcal{T}_H . Let $Q_H : L^2(\Omega) \mapsto \bar{V}_H$ be the L^2 -projection. Finally, introduce also the subspace V_H of \bar{V}_H of functions that have zero mean value over Ω , i.e., $V_H \subset L_0^2(\Omega)$. We note that $Q_H : L_0^2(\Omega) \mapsto V_H$, that is, if f has zero meanvalue then $Q_H f$ does too.

The following commutativity property holds

$$\text{div } \mathbf{\Pi}_H \mathbf{g} = Q_H \text{div } \mathbf{g}.$$

Given a sufficiently smooth function \mathbf{g} , the above equality tells us that there is $\boldsymbol{\psi}_H \in \mathbf{R}_H$, $\boldsymbol{\psi}_H = \mathbf{\Pi}_H \mathbf{g}$, which satisfies $\text{div } \boldsymbol{\psi}_H = Q_H \text{div } \mathbf{g}$ and therefore $\|\text{div } \boldsymbol{\psi}_H\|_0 = \|\text{div } \mathbf{g}\|_0$. The main result of this section is the following stability estimate, which addresses the case when \mathbf{g} is a general $H(\text{div})$ function.

Lemma 2.1. *Given $\mathbf{g} \in \mathbf{H}_0(\Omega, \text{div})$, there is a $\boldsymbol{\psi}_H \in \mathbf{R}_H$ such that $\text{div } \boldsymbol{\psi}_H = Q_H \text{div } \mathbf{g}$ and*

$$(2.2) \quad \|\boldsymbol{\psi}_H\|_0^2 \leq C (\|\mathbf{g}\|_0^2 + H^2 \|\text{div } \mathbf{g}\|_0^2).$$

Proof. We use the construction from [Va08], p. 500 leading to estimate (F.27) there. For completeness, we provide the corresponding details.

Let p be the solution of the Neumann problem,

$$-\Delta_N p = Q_H \text{div } \mathbf{g}.$$

Using regularity, we have that $\nabla p \in \mathbf{H}^{\frac{1}{2}+\delta}(\Omega)$, that is the normal trace $\nabla p \cdot \mathbf{n}$ belongs to $H^\delta(F)$ for any straight line/planar surface F contained in $\bar{\Omega}$. This shows that $\boldsymbol{\psi}_H = \mathbf{\Pi}_H(-\nabla p) \in \mathbf{R}_H$ is well-defined. It satisfies

$$\text{div } \boldsymbol{\psi}_H = -\text{div}(\mathbf{\Pi}_H \nabla p) = -Q_H \text{div } \nabla p = -Q_H \Delta_N p = Q_H \text{div } \mathbf{g}.$$

Next, we use the L_2 -approximation property (2.1) which combined with the assumed regularity estimate and an inverse inequality, shows

$$(2.3) \quad \begin{aligned} \|\boldsymbol{\psi}_H + \nabla p\|_0 &= \|(I - \mathbf{\Pi}_H) \nabla p\|_0 \\ &\leq CH^{\frac{1}{2}+\delta} \|\nabla p\|_{\frac{1}{2}+\delta} \\ &\leq CH^{\frac{1}{2}+\delta} \|p\|_{\frac{3}{2}+\delta} \\ &\leq CH^{\frac{1}{2}+\delta} \|Q_H \text{div } \mathbf{g}\|_{-\frac{1}{2}+\delta} \\ &= CH^{\frac{1}{2}+\delta} \|Q_H \text{div } \mathbf{g}\|_{-1+(\frac{1}{2}+\delta)} \\ &\leq C \|Q_H \text{div } \mathbf{g}\|_{-1} \\ &\leq C (\|\text{div } \mathbf{g}\|_{-1} + \|(I - Q_H) \text{div } \mathbf{g}\|_{-1}) \\ &\leq C (\|\mathbf{g}\|_0 + H \|\text{div } \mathbf{g}\|_0). \end{aligned}$$

We used the approximation property of the discontinuous (piecewise constant) projection Q_H in $H^{-1}(\Omega)$, which is the dual space of $H_0^1(\Omega)$. Specifically, for $f = \text{div } \mathbf{g} \in$

$L_0^2(\Omega)$, we use the estimate

$$\begin{aligned} \|(I - Q_H)f\|_{-1} &= \sup_{\phi \in H_0^1(\Omega)} \frac{((I - Q_H)f, \phi)}{\|\phi\|_1} \\ &= \sup_{\phi \in H_0^1(\Omega)} \frac{(f, (I - Q_H)\phi)}{\|\phi\|_1} \\ &\leq CH \|f\|_0. \end{aligned}$$

We also have $\|\nabla p\|_0^2 = (Q_H \operatorname{div} \mathbf{g}, p) = -(\operatorname{div} \mathbf{g}, (I - Q_H)p) + (p, \operatorname{div} \mathbf{g}) = -(\operatorname{div} \mathbf{g}, (I - Q_H)p) - (\nabla p, \mathbf{g})$, which shows

$$\|\nabla p\|_0 \leq C (H \|\operatorname{div} \mathbf{g}\|_0 + \|\mathbf{g}\|_0).$$

This estimate combined with (2.3) used in the triangle inequality

$$\|\boldsymbol{\psi}_H\|_0 \leq \|\boldsymbol{\psi}_H + \nabla p\|_0 + \|\nabla p\|_0 \leq C (\|\mathbf{g}\|_0 + H \|\operatorname{div} \mathbf{g}\|_0),$$

implies the desired bound (2.2). \square

3. PROOF OF THEOREM 1.2

In this section we prove the equivalence of (1.2) and (1.4) as stated in Theorem 1.2. First, we show the following auxiliary result that characterizes the left-hand side of the inf-sup condition (1.4).

Lemma 3.1. *For any $\bar{p} \in L_0^2(\Omega)$, the following equality holds:*

$$(3.1) \quad \left((\tau I - \Delta_N^{-1})^{-1} \bar{p}, \bar{p} \right)^{\frac{1}{2}} = \sup_{\mathbf{u} \in \mathbf{H}_0(\Omega, \operatorname{div})} \frac{(\bar{p}, \operatorname{div} \mathbf{u})}{(\|\mathbf{u}\|_0^2 + \tau \|\operatorname{div} \mathbf{u}\|_0^2)^{\frac{1}{2}}}.$$

Proof. Given $\bar{p} \in L_0^2(\Omega)$ let $\bar{\mathbf{u}}$ be the unique solution of the equation

$$(\bar{\mathbf{u}}, \mathbf{v}) + \tau(\operatorname{div} \bar{\mathbf{u}}, \operatorname{div} \mathbf{v}) = (\bar{p}, \operatorname{div} \mathbf{v}) \forall \mathbf{v} \in \mathbf{H}_0(\Omega, \operatorname{div}).$$

Recalling the definition of $\mathcal{L}_\tau^{\operatorname{div}}$ in (1.1) we notice that

$$(3.2) \quad \sup_{\mathbf{u} \in \mathbf{H}_0(\Omega, \operatorname{div})} \frac{(\bar{p}, \operatorname{div} \mathbf{u})}{(\|\mathbf{u}\|_0^2 + \tau \|\operatorname{div} \mathbf{u}\|_0^2)^{\frac{1}{2}}} = \sup_{\mathbf{u}} \frac{\mathcal{L}_\tau^{\operatorname{div}}(\bar{\mathbf{u}}, \mathbf{u})}{\|\mathbf{u}\|_{\mathcal{L}_\tau^{\operatorname{div}}}} = \|\bar{\mathbf{u}}\|_{\mathcal{L}_\tau^{\operatorname{div}}} = \sqrt{(\bar{p}, \operatorname{div} \bar{\mathbf{u}})}.$$

To complete the proof it remains to show that

$$(3.3) \quad \operatorname{div} \bar{\mathbf{u}} = (\tau I - \Delta_N^{-1})^{-1} \bar{p}.$$

Let $\bar{q} \in L_0^2(\Omega)$ be arbitrary, and $\psi \in H^1(\Omega) \cap L_0^2(\Omega)$ be the solution of $-\Delta_N \psi = \bar{q}$. Then

$$(3.4) \quad \|\Delta_N^{-1} \bar{q}\|_0 = \|\psi\|_0 \leq \sqrt{C_P} \|\nabla \psi\|_0 \leq C_P \|\bar{q}\|_0.$$

where C_P is the constant from the Poincaré inequality $\|\psi\|_0^2 \leq C_P \|\nabla \psi\|_0^2$. This means that $(-\Delta_N)^{-1} + \tau I : L_0^2(\Omega) \mapsto L_0^2(\Omega)$ is bounded and coercive, and hence invertible operator, i.e. $(\tau I - \Delta_N^{-1})^{-1} \bar{p}$ is well-defined.

Set $\mathbf{v} = -\nabla \psi = \nabla \Delta_N^{-1} \bar{q}$. By definition $\mathbf{v} \in \mathbf{H}_0(\Omega, \operatorname{div})$ with $\operatorname{div} \mathbf{v} = \bar{q}$. Testing the $\mathcal{L}_\tau^{\operatorname{div}}$ form with this function we get

$$(\bar{\mathbf{u}}, \nabla \Delta_N^{-1} \bar{q}) + \tau (\operatorname{div} \bar{\mathbf{u}}, \bar{q}) = (\bar{p}, \bar{q}),$$

which after integration-by-parts and the use of symmetry of $-\Delta_N^{-1}$, results in

$$(-\Delta_N^{-1} \text{div } \bar{\mathbf{u}}, \bar{q}) + \tau (\text{div } \bar{\mathbf{u}}, \bar{q}) = (\bar{p}, \bar{q}), \forall \bar{q} \in L_0^2(\Omega).$$

That is, $\bar{\theta} = \text{div } \bar{\mathbf{u}} \in L_0^2(\Omega)$ solves the equation $((-\Delta_N)^{-1} + \tau I) \bar{\theta} = \bar{p}$ which concludes the proof. \square

Proof of Theorem 1.2. Assume first that (1.4) holds. Given $\mathbf{u} \in \mathbf{H}_0(\Omega, \text{div})$, consider the parameter-dependent problem (1.3). Using the first equation of (1.3) in the inf-sup estimate (1.4), implies

$$(3.5) \quad \left((\tau I - \Delta_N^{-1})^{-1} \bar{p}, \bar{p} \right)^{\frac{1}{2}} \leq \frac{1}{c_0} \left(\|\mathbf{z}\|_0^2 + \tau \|\nabla \mathbf{z}\|_0^2 \right)^{\frac{1}{2}}.$$

Using again the first equation of (1.3) for $\boldsymbol{\theta} = \mathbf{z}$, the fact that $\text{div } \mathbf{z} = \text{div } \mathbf{u}$ and the inequality

$$-(\bar{p}, \text{div } \mathbf{u}) \leq \left((\tau I - \Delta_N^{-1})^{-1} \bar{p}, \bar{p} \right)^{\frac{1}{2}} \left((\tau I - \Delta_N^{-1}) \text{div } \mathbf{u}, \text{div } \mathbf{u} \right)^{\frac{1}{2}},$$

combined with (3.5), gives

$$(\mathcal{L}_\tau \mathbf{z}, \mathbf{z}) \leq \frac{1}{c_0} \left(\|\mathbf{z}\|_0^2 + \tau \|\nabla \mathbf{z}\|_0^2 \right)^{\frac{1}{2}} \left((\tau I - \Delta_N^{-1}) \text{div } \mathbf{u}, \text{div } \mathbf{u} \right)^{\frac{1}{2}}.$$

Let $\psi \in H^1(\Omega) \cap L_0^2(\Omega)$ solve the problem $-\Delta_N \psi = \text{div } \mathbf{u}$. We have

$$(\text{div } \mathbf{u}, \psi) = (\nabla \psi, \nabla \psi) \leq \|\mathbf{u}\|_0^2.$$

Therefore,

$$\left((\tau I - \Delta_N^{-1}) \text{div } \mathbf{u}, \text{div } \mathbf{u} \right)^{\frac{1}{2}} = \left(\tau \|\text{div } \mathbf{u}\|_0^2 + (\text{div } \mathbf{u}, \psi) \right)^{\frac{1}{2}} \leq \left(\tau \|\text{div } \mathbf{u}\|_0^2 + \|\mathbf{u}\|_0^2 \right)^{\frac{1}{2}}.$$

That is, we have

$$(\mathcal{L}_\tau \mathbf{z}, \mathbf{z}) \leq \frac{1}{c_0} \left(\|\mathbf{z}\|_0^2 + \tau \|\nabla \mathbf{z}\|_0^2 \right)^{\frac{1}{2}} \left(\tau \|\text{div } \mathbf{u}\|_0^2 + \|\mathbf{u}\|_0^2 \right)^{\frac{1}{2}},$$

which is the desired stability result (1.2).

Now, we prove the converse statement. Given $\mathbf{u} \in \mathbf{H}_0(\Omega, \text{div})$, let $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$ be such that $\text{div } \mathbf{z} = \text{div } \mathbf{u}$ and

$$c_0 \|\mathbf{z}\|_{\mathcal{L}_\tau} \leq \|\mathbf{u}\|_{\mathcal{L}_\tau^{\text{div}}}$$

Then (3.1) implies

$$\left((\tau I - \Delta_N^{-1})^{-1} \bar{p}, \bar{p} \right)^{\frac{1}{2}} = \sup_{\mathbf{u} \in \mathbf{H}_0(\Omega, \text{div})} \frac{(\bar{p}, \text{div } \mathbf{u})}{\|\mathbf{u}\|_{\mathcal{L}_\tau^{\text{div}}}} \leq \frac{1}{c_0} \sup_{\mathbf{z} \in \mathbf{H}_0^1(\Omega)} \frac{(\bar{p}, \text{div } \mathbf{z})}{\|\mathbf{z}\|_{\mathcal{L}_\tau}},$$

which is the desired result. \square

In the following two sections, we concentrate on establishing Theorem 1.1 in the more difficult (for the analysis) case $0 < \tau \leq C$. The case of ‘‘large’’ τ will be considered in Section 6. Here, ‘‘large’’ refers to values of τ which are (much) bigger than the diameter of Ω .

4. AN “INF-SUP” RESULT FOR PARAMETER-DEPENDENT NORMS

As stated in the introduction, the inf-sup condition (1.4) (and therefore the regular decomposition (1.2)) is well-known for convex domains. Since any polygonal/polyhedral Lipschitz domain can be decomposed into a finite union of convex domains, it is enough to assume that Ω can be represented as union of two domains for each of which (1.4) holds, and prove that Theorem 1.1 is also satisfied for Ω . This is done in the current section under an additional assumption (A) stated below. We verify the assumption (A) in the following section.

Let $\Omega = \Omega_1 \cup \Omega_2$ and assume that we have the inf-sup condition on each of Ω_i . Furthermore, we make the following key assumption:

- (A) For each $\mathbf{u} \in \mathbf{H}_0(\Omega, \text{div})$ and any parameter $\tau > 0$, there is a decomposition of $\text{div } \mathbf{u}$, $\text{div } \mathbf{u} = \text{div } \mathbf{u}_1 + \text{div } \mathbf{u}_2$ with $\mathbf{u}_i \in \mathbf{H}_0(\Omega_i, \text{div})$ which is stable in the sense that

$$(4.1) \quad ((\tau I - \Delta_N^{-1}) \text{div } \mathbf{u}_1, \text{div } \mathbf{u}_1) + ((\tau I - \Delta_N^{-1}) \text{div } \mathbf{u}_2, \text{div } \mathbf{u}_2) \leq c_1^2 \|\mathbf{u}\|_{\mathcal{L}^{\text{div}}}^2.$$

For a fixed $\mathbf{u} \in \mathbf{H}_0(\Omega, \text{div})$ and a positive parameter $\tau = \mathcal{O}(1)$, consider the saddle-point problem: Find $\mathbf{z}_i \in \mathbf{H}_0^1(\Omega_i)$, $i = 1, 2$ and $p \in L_0^2(\Omega)$ such that

$$(4.2) \quad \begin{aligned} (\mathcal{L}_\tau \mathbf{z}_1, \mathbf{v}_1) &+ (p, \text{div } \mathbf{v}_1) = 0, & \forall \mathbf{v}_1 \in \mathbf{H}_0^1(\Omega_1), \\ (\mathcal{L}_\tau \mathbf{z}_2, \mathbf{v}_2) &+ (p, \text{div } \mathbf{v}_2) = 0, & \forall \mathbf{v}_2 \in \mathbf{H}_0^1(\Omega_2), \\ (\text{div}(\mathbf{z}_1 + \mathbf{z}_2), q) &= (\text{div } \mathbf{u}, q), & \forall q \in L_0^2(\Omega). \end{aligned}$$

In the last equation, we define each of \mathbf{z}_i on the whole of Ω using a simple extension by zero. Introducing $q_i = q|_{\Omega_i} - \frac{1}{|\Omega_i|} \int_{\Omega_i} q(\mathbf{x}) d\mathbf{x} \in L_0^2(\Omega_i)$ for any $q \in L^2(\Omega)$, the above

problem can be rewritten as

$$\begin{aligned} (\mathcal{L}_\tau \mathbf{z}_1, \mathbf{v}_1) &+ (p_1, \text{div } \mathbf{v}_1) = 0, & \forall \mathbf{v}_1 \in \mathbf{H}_0^1(\Omega_1), \\ (\mathcal{L}_\tau \mathbf{z}_2, \mathbf{v}_2) &+ (p_2, \text{div } \mathbf{v}_2) = 0, & \forall \mathbf{v}_2 \in \mathbf{H}_0^1(\Omega_2), \\ (\text{div } \mathbf{z}_1, q_1) &+ (\text{div } \mathbf{z}_2, q_2) = (\text{div } \mathbf{u}, q), & \forall q \in L_0^2(\Omega). \end{aligned}$$

Let $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ where $\mathbf{u}_i \in \mathbf{H}_0(\Omega_i, \text{div})$. Then

$$(\text{div } \mathbf{u}, q) = (\text{div } \mathbf{u}_1, q_1) + (\text{div } \mathbf{u}_2, q_2).$$

Introduce the invertible operators $\mathcal{L}_{\tau, \Omega_i} : \mathbf{H}_0^1(\Omega_i) \mapsto \mathbf{H}_0^1(\Omega_i)$ as the restrictions of \mathcal{L}_τ to $\mathbf{H}_0^1(\Omega_i)$. After eliminating \mathbf{z}_i , we get the following reduced problem, letting $S_i = \text{div } \mathcal{L}_{\tau, \Omega_i}^{-1} \text{div}^* : L_0^2(\Omega_i) \mapsto L_0^2(\Omega_i)$,

$$(S_1 p_1, q_1) + (S_2 p_2, q_2) = -(\text{div } \mathbf{u}_1, q_1) - (\text{div } \mathbf{u}_2, q_2).$$

Next, using the following estimate, which is equivalent with the inf-sup condition (1.4),

$$c_0((\tau I - \Delta_N^{-1})^{-1} p_i, p_i) \leq (S_i p_i, p_i),$$

we obtain, for $q = p$,

$$\begin{aligned} &c_0((\tau I - \Delta_N^{-1})^{-1} p_1, p_1) + ((\tau I - \Delta_N^{-1})^{-1} p_2, p_2) \\ &\leq |(\text{div } \mathbf{u}_1, p_1)| + |(\text{div } \mathbf{u}_2, p_2)| \\ &\leq (((\tau I - \Delta_N^{-1}) \text{div } \mathbf{u}_1, \text{div } \mathbf{u}_1) + ((\tau I - \Delta_N^{-1}) \text{div } \mathbf{u}_2, \text{div } \mathbf{u}_2))^{\frac{1}{2}} \\ &\quad \times (((\tau I - \Delta_N^{-1})^{-1} p_1, p_1) + ((\tau I - \Delta_N^{-1})^{-1} p_2, p_2))^{\frac{1}{2}}, \end{aligned}$$

and arrive at the a priori estimate for p ,

$$\begin{aligned} & c_0^2 \left(((\tau I - \Delta_N^{-1})^{-1} p_1, p_1) + ((\tau I - \Delta_N^{-1})^{-1} p_2, p_2) \right) \\ & \leq ((\tau I - \Delta_N^{-1}) \text{div } \mathbf{u}_1, \text{div } \mathbf{u}_1) + ((\tau I - \Delta_N^{-1}) \text{div } \mathbf{u}_2, \text{div } \mathbf{u}_2). \end{aligned}$$

Adding the first two equations in (4.2) with $\mathbf{v}_i = \mathbf{z}_i$, and using the last equation (i.e., $\text{div}(\mathbf{z}_1 + \mathbf{z}_2) = \text{div } \mathbf{u}$), based on assumption (A), we obtain

$$\begin{aligned} (\mathcal{L}_\tau \mathbf{z}_1, \mathbf{z}_1) + (\mathcal{L}_\tau \mathbf{z}_2, \mathbf{z}_2) &= -(p, \text{div}(\mathbf{z}_1 + \mathbf{z}_2)) = -(p, \text{div}(\mathbf{u}_1 + \mathbf{u}_2)) \\ &= -(p_1, \text{div } \mathbf{u}_1) - (p_2, \text{div } \mathbf{u}_2) \\ &\leq \left(((\tau I - \Delta_N^{-1})^{-1} p_1, p_1) + ((\tau I - \Delta_N^{-1})^{-1} p_2, p_2) \right)^{\frac{1}{2}} \\ &\quad \times \left(((\tau I - \Delta_N^{-1}) \text{div } \mathbf{u}_1, \text{div } \mathbf{u}_1) + ((\tau I - \Delta_N^{-1}) \text{div } \mathbf{u}_2, \text{div } \mathbf{u}_2) \right)^{\frac{1}{2}} \\ &\leq \frac{1}{c_0} \left(((\tau I - \Delta_N^{-1}) \text{div } \mathbf{u}_1, \text{div } \mathbf{u}_1) + ((\tau I - \Delta_N^{-1}) \text{div } \mathbf{u}_2, \text{div } \mathbf{u}_2) \right) \\ &\leq \frac{c_1}{c_0} (\|\mathbf{u}\|_0^2 + \tau \|\text{div } \mathbf{u}\|_0^2). \end{aligned}$$

Therefore, letting $\mathbf{z} = \mathbf{z}_1 + \mathbf{z}_2 \in \mathbf{H}_0^1(\Omega)$, we have $\text{div } \mathbf{z} = \text{div } \mathbf{u}$ and

$$(\mathcal{L}_\tau \mathbf{z}, \mathbf{z}) \leq 2 \frac{c_1^2}{c_0^2} (\|\mathbf{u}\|_0^2 + \tau \|\text{div } \mathbf{u}\|_0^2).$$

In conclusion, we have proved the following main result.

Theorem 4.1. *Let $\Omega = \Omega_1 \cup \Omega_2$ be union of two overlapping subdomains for which a stable decomposition for $\mathbf{H}_0(\Omega, \text{div})$ satisfying (4.1) exists. The domains Ω_i are also such that Theorem 1.1 holds for them. Then, Theorem 1.1 holds for Ω .*

5. STABLE DECOMPOSITION OF FUNCTIONS IN $\mathbf{H}_0(\Omega, \text{div})$ WITH PROPERTY (A)

In this section, we verify the key assumption (A).

Let \mathcal{T}_H be an auxiliary mesh on Ω of triangular or tetrahedral elements of mesh size H . Such a mesh exists because Ω is polygonal/polyhedral domain. Let \mathbf{R}_H be the associated Raviart–Thomas finite element space as in Section 2. We assume that

$$(5.1) \quad c\tau \leq H^2 \leq C\tau,$$

where $\tau > 0$ is our given (possibly small) parameter. Values of τ that do not satisfy this assumption will be addressed in the following section.

Let $\mathbf{u} \in \mathbf{H}_0(\Omega, \text{div})$. Consider $(I - Q_H) \text{div } \mathbf{u}$, where Q_H is the $L^2(\Omega)$ projection onto V_H defined in Section 2. We note that for any subdomain D that is exactly covered by elements from \mathcal{T}_H , we have $(I - Q_H) \text{div } \mathbf{u}|_D \in L_0^2(D)$, that is, this function has zero mean value over D . This is in particular true (by assumption) for $D = \Omega_1$ and $D = \Omega_2 \setminus \Omega_1 = \Omega \setminus \Omega_1$.

We first solve the Neumann problem for the Laplacian $-\Delta_N \psi_1 = (I - Q_H) \text{div } \mathbf{u}$ in Ω_1 . We have $\|\nabla \psi_1\|_0^2 = ((I - Q_H) \text{div } \mathbf{u}, \psi_1) = (\text{div } \mathbf{u}, (I - Q_H) \psi_1) \leq \|\text{div } \mathbf{u}\|_0 \|(I - Q_H) \psi_1\|_0 \leq CH \|\text{div } \mathbf{u}\|_0 \|\nabla \psi_1\|_0$. That is, for $\mathbf{g}_1 = \nabla \psi_1 \in \mathbf{H}_0(\Omega_1, \text{div})$, we have

$$\text{div } \mathbf{g}_1 = (I - Q_H) \text{div } \mathbf{u} \quad \text{on } \Omega_1, \quad \text{and} \quad \|\mathbf{g}_1\|_0^2 \leq CH^2 \|\text{div } \mathbf{u}\|_0^2,$$

which shows the first stability result. Similarly, we can solve the Neumann problem for the Laplacian $-\Delta_N \psi_2 = (I - Q_H) \text{div } \mathbf{u}$ in $\Omega_2 \setminus \Omega_1$. Then, for $\mathbf{g}_2 = \nabla \psi_2 \in \mathbf{H}_0(\Omega_2 \setminus \Omega_1, \text{div})$, we have

$$\text{div } \mathbf{g}_2 = (I - Q_H) \text{div } \mathbf{u} \quad \text{on } \Omega_2 \setminus \Omega_1, \quad \text{and} \quad \|\mathbf{g}_2\|_0^2 \leq CH^2 \|\text{div } \mathbf{u}\|_0^2.$$

In conclusion, we can find $\mathbf{g}_i \in \mathbf{H}_0(\Omega, \text{div})$ supported in the respective subdomains Ω_i such that

- (i) $\text{div}(\sum_i \mathbf{g}_i) = (I - Q_H) \text{div} \mathbf{u}$, and
- (ii) $\sum_i (\|\mathbf{g}_i\|_0^2 + \tau \|\text{div} \mathbf{g}_i\|_0^2) \leq C (\tau + H^2) \|\text{div} \mathbf{u}\|_0^2$.

Next, we decompose the piecewise-constant function $Q_H \text{div} \mathbf{u}$. Note that this function has zero mean value over Ω . Based on Lemma 2.1, there is a $\mathbf{u}_H \in \mathbf{R}_H$ such that $\text{div} \mathbf{u}_H = Q_H \text{div} \mathbf{u}$ which satisfies the stability estimate

$$(5.2) \quad \|\mathbf{u}_H\|_0^2 + \tau \|\text{div} \mathbf{u}_H\|_0^2 \leq c_0^2 (\|\mathbf{u}\|_0^2 + (\tau + H^2) \|\text{div} \mathbf{u}\|_0^2).$$

Now, let us split \mathbf{u}_H into two components $\mathbf{u}_H^{(1)}$ and $\mathbf{u}_H^{(2)}$. Let $\{\Phi_F^{(H)}\}_{F \in \mathcal{F}}$ be the set of basis functions in \mathbf{R}_H . Here, \mathcal{F} is the set of interior (to Ω) faces of elements in \mathcal{T}_H . Then, let

$$\mathbf{u}_H = \sum_{F \in \mathcal{F}} u_F \Phi_F^{(H)}.$$

To define the splitting, we can proceed as follows. Let $\Omega_0 = \Omega_1 \cap \Omega_2$. Let \mathcal{F}_0 be the set of element faces that are interior to Ω_0 . Define

$$\mathbf{u}_H^{(0)} = \sum_{F \in \mathcal{F}_0} u_F \Phi_F^{(H)}.$$

Note that $\mathbf{u}_H^{(0)}$ has zero normal trace on $\partial\Omega_1$ and $\partial\Omega_2$. Similarly, let \mathcal{F}_i be the set of element faces interior to Ω_i . Define

$$\mathbf{u}_H^{(i)} = \sum_{F \in \mathcal{F}_i \setminus \mathcal{F}_0} u_F \Phi_F^{(H)} + \frac{1}{2} \mathbf{u}_H^{(0)}.$$

We have $\mathbf{u}_H = \mathbf{u}_H^{(1)} + \mathbf{u}_H^{(2)}$ and $\|\mathbf{u}_H^{(i)}\|_0 \leq C \|\mathbf{u}_H\|_0$. Also, by construction $\mathbf{u}_H^{(i)}$ has zero normal trace on $\partial\Omega_i$ and vanishes outside $\bar{\Omega}_i$, that is $\mathbf{u}_H^{(i)} \in \mathbf{H}_0(\Omega_i, \text{div})$. Finally, using inverse inequality, we have

$$\|\text{div} \mathbf{u}_H^{(i)}\|_0 \leq CH^{-1} \|\mathbf{u}_H^{(i)}\|_0 \leq CH^{-1} \|\mathbf{u}_H\|_0.$$

In conclusion, based on the above results combined with (5.2), we have $\mathbf{u}_H = \mathbf{u}_H^{(1)} + \mathbf{u}_H^{(2)}$, where each $\mathbf{u}_H^{(i)}$ is supported in the respective subdomain Ω_i , and the following stability estimates hold:

$$(5.3) \quad \begin{aligned} ((\tau I - \Delta_N^{-1}) \text{div} \mathbf{u}_H^{(i)}, \text{div} \mathbf{u}_H^{(i)}) &\leq \|\mathbf{u}_H^{(i)}\|_0^2 + \tau \|\text{div} \mathbf{u}_H^{(i)}\|_0^2 \\ &\leq C \left(1 + \frac{\tau}{H^2}\right) \|\mathbf{u}_H\|_0^2 \\ &\leq C \left(1 + \frac{\tau}{H^2}\right) (\|\mathbf{u}\|_0^2 + H^2 \|\text{div} \mathbf{u}\|_0^2) \\ &= C \left[\left(1 + \frac{\tau}{H^2}\right) \|\mathbf{u}\|_0^2 + (\tau + H^2) \|\text{div} \mathbf{u}\|_0^2\right]. \end{aligned}$$

We recall that we have assumed that $\frac{\tau}{H^2} + \frac{H^2}{\tau} = \mathcal{O}(1)$ (see (5.1)).

The final stable decomposition is defined by the components $\mathbf{g}_i + \mathbf{u}_H^{(i)} \in \mathbf{H}_0(\Omega_i, \text{div})$ (see (i) and (ii) for \mathbf{g}_i). This completes the verification of assumption (A).

Remark 5.1. *The following decomposition result for $\mathbf{H}_0(\Omega, \text{div})$ holds. Based on the proved property (A), we have that for any $\mathbf{u} \in \mathbf{H}_0(\Omega, \text{div})$ there are stable components $\bar{\mathbf{u}}_i \in \mathbf{H}_0(\Omega_i, \text{div})$ such that*

$$\text{div}(\mathbf{u} - \sum_i \bar{\mathbf{u}}_i) = 0.$$

This means, assuming for simplicity that Ω is simply connected with simply connected boundary, that there is a $\boldsymbol{\zeta} \in \mathbf{H}_0(\text{curl}, \Omega)$ (see e.g., [GR86], [Mo03]) such that

$$\mathbf{u} = \sum_i \bar{\mathbf{u}}_i + \text{curl } \boldsymbol{\zeta}.$$

Based on a regular decomposition result (cf., e.g. [KV09]), we may assume that $\boldsymbol{\zeta} \in \mathbf{H}_0^1(\Omega)$ with

$$\|\boldsymbol{\zeta}\|_1 \leq C \|\mathbf{u} - \sum_i \bar{\mathbf{u}}_i\|_0.$$

Finally, using a stable decomposition for the \mathbf{H}_0^1 conforming space, (for a proof of this classical result, see, e.g., [Va08], pp. 473-475)

$$\boldsymbol{\zeta} = \sum_i \boldsymbol{\zeta}_i,$$

where $\boldsymbol{\zeta}_i \in \mathbf{H}_0^1(\Omega_i)$ such that

$$\sum_i \|\nabla \boldsymbol{\zeta}_i\|_0^2 \leq C \|\nabla \boldsymbol{\zeta}\|_0^2,$$

we have the decomposition

$$\mathbf{u} = \sum_i \mathbf{u}_i = \sum_i (\bar{\mathbf{u}}_i + \text{curl } \boldsymbol{\zeta}_i).$$

Note that now each $\mathbf{u}_i = \bar{\mathbf{u}}_i + \text{curl } \boldsymbol{\zeta}_i \in \mathbf{H}_0(\text{div}, \Omega_i)$ and the following stability estimate holds:

$$\sum_i \|\mathbf{u}_i\|_0^2 + \tau \sum_i \|\text{div } \mathbf{u}_i\|_0^2 \leq C (\|\mathbf{u}\|_0^2 + \tau \|\text{div } \mathbf{u}\|_0^2).$$

6. REGULAR DECOMPOSITION FOR THE CASE OF LARGE τ

Note that estimate (5.3) does not work if τ is large, i.e., (much) bigger than the diameter of Ω . For smaller τ , we have the flexibility to choose H comparable to τ . For large τ this is not possible.

In this section we prove Theorem 1.1 directly in the remaining case

$$(6.1) \quad \frac{1}{\tau} \leq C.$$

Specifically, the existence of $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$ such that $\text{div } \mathbf{z} = \text{div } \mathbf{u}$ satisfying the uniform estimate

$$c_0^2 \mathcal{L}_\tau(\mathbf{z}, \mathbf{z}) \leq \|\mathbf{u}\|_0^2 + \tau \|\text{div } \mathbf{u}\|_0^2,$$

follows from the standard “inf-sup” estimate for Stokes problem,

$$c_0 \|p\|_0 \leq \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{(p, \text{div } \mathbf{v})}{\|\mathbf{v}\|_1}.$$

For a detailed proof of this estimate for general Lipschitz domains, cf., e.g., [Br03]. The existence of the required \mathbf{z} is provided by the solution \mathbf{z} of (1.3). Indeed for large τ (as in (6.1)), the above inf-sup estimate implies that

$$c_0 \|\bar{p}\|_0 \leq \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{\mathcal{L}_\tau(\mathbf{z}, \mathbf{v})}{\|\mathbf{v}\|_1} \leq C\tau \|\mathbf{z}\|_1.$$

Hence

$$(\mathcal{L}_\tau \mathbf{z}, \mathbf{z}) = -(\bar{p}, \operatorname{div} \mathbf{z}) = -(\bar{p}, \operatorname{div} \mathbf{u}) \leq C\tau \|\mathbf{z}\|_1 \|\operatorname{div} \mathbf{u}\|_0 \leq C \|\mathbf{z}\|_{\mathcal{L}_\tau} \|\mathbf{u}\|_{\mathcal{L}_\tau^{\operatorname{div}}}$$

That is, we have the desired stability estimate

$$(\mathcal{L}_\tau \mathbf{z}, \mathbf{z}) \leq C \mathcal{L}_\tau^{\operatorname{div}}(\mathbf{u}, \mathbf{u}).$$

Note that using this result in Theorem 1.2 shows also the parameter-dependent “inf-sup” estimate (1.4) for large τ (as in (6.1)). The latter is actually seen directly, since for large τ , using the boundedness of $-\Delta_N^{-1}$ (see (3.4)), the Poincaré inequality $\|\mathbf{v}\|_0^2 \leq C_P \|\nabla \mathbf{v}\|_0^2$ and (6.1), we have

$$\begin{aligned} \left((\tau I - \Delta_N^{-1})^{-1} p, p \right) &\simeq \frac{1}{\tau} \|p\|_0^2 \leq \frac{1}{c_0^2} \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{(p, \operatorname{div} \mathbf{v})^2}{\tau \|\mathbf{v}\|_1^2} \\ &\leq \frac{1}{c_0^2} \left(1 + \frac{C_P}{\tau}\right) \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{(p, \operatorname{div} \mathbf{v})^2}{(\mathcal{L}_\tau \mathbf{v}, \mathbf{v})} \\ &\leq \frac{1}{c_1^2} \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{(p, \operatorname{div} \mathbf{v})^2}{(\mathcal{L}_\tau \mathbf{v}, \mathbf{v})}. \end{aligned}$$

Combining the results of the last three sections we conclude that Theorem 1.1 holds for general Lipschitz polygonal/polyhedral Ω and general positive parameters τ .

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