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On two-grid convergence estimates

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The paper is dedicated to the 70th birthday of Owe Axelsson, a pioneer in two–level preconditioning methods.

SUMMARY

We derive a new representation for the exact convergence factor of classical two-level and two-grid preconditioners. Based on this result, we establish necessary and sufficient conditions for constructing the components of efficient algebraic multigrid (AMG) methods. The relation of the sharp estimate to the classical two-level hierarchical basis methods is discussed as well. Lastly, as an application, we give an optimal two-grid convergence proof of a purely algebraic "window"-AMG method. Copyright © 2000 John Wiley & Sons, Ltd.

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1. Introduction

The primary objective of this paper is the presentation of a new sharp convergence theory for both two-level hierarchical basis (TL) methods and two-grid (TG) methods. Our focus is on the possible impact of the new theory in algebraic multigrid (AMG) methods.

The main result we prove is Theorem 4.1, which we then tailor to the respective cases of TL and TG methods in Theorems 4.2 and 4.3. For TL methods, Theorem 4.2 deals also with inexact coarse matrices. It generalizes a main two-level convergence theorem in [2]. Also, it leads to a well–known classical convergence estimate in terms of $\cos^2$ of the abstract angle between the two hierarchical component spaces. For TG methods, of particular interest is the relationship of the sharp result in Theorem 4.3 with the (non-sharp) theory recently developed

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in [12]. The latter theory introduces a weak approximation property that accounts for general smoothing and coarsening processes. It also motivates the use of so-called *compatible relaxation* [7] as a technique for efficiently measuring the quality of a coarse grid, and hence, as a tool for selecting coarse grids in AMG methods. An important component of the theory is the use of a restriction-like operator R that defines the *coarse-grid variables*. Curiously, the expression in (4.9) of Theorem 4.3 is exactly the same as the expression used in [12], but for a specific R. Since the construction of compatible relaxation is based on R, one possible outcome of the relationship between these theories is the development of new, more predictive, compatible relaxation methods. In general, it is hoped that the theory presented here will serve as a guide for many future algorithm developments.

The remainder of the paper is structured as follows. In Section 2, we provide some basic definitions, inequalities, and results needed throughout the paper. In Section 3, we prove an important auxiliary result, the so-called "saddle-point lemma", and then in Section 4, we use this lemma to prove the main (sharp) two-grid convergence result. Next, in Section 5, we relate the sharp two-grid convergence result with existing tools commonly used to prove two-grid convergence in AMG theory. Many facts in Section 5 are simply reformulation of results already established in a previous paper [12], but here we are able to establish some additional necessary conditions. Finally, in Section 6 we show how the presented sharp convergence result can be used to derive upper bounds for the two-grid convergence rate of a so-called "window"-based spectral AMG method (a variant of the method proposed in Chartier et al. [9]).

2. Preliminaries

This paper is concerned with two-level hierarchical basis (TL) or more generally with two-grid (TG) methods. We consider a vector space V, isomorphic to $\mathbb{R}^n$ for some n. The space V is equipped with the usual Euclidean vector inner product $\langle \mathbf{v},\ \mathbf{w}\rangle = \mathbf{w}^T\mathbf{v}$. Our focus will be on two-level or two-grid iterative methods for the solution of

$$A\mathbf{u} = \mathbf{f}, \tag{2.1}$$

where $A : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a *symmetric* and *positive definite* (s.p.d.) matrix. A linear iterative method takes the form: Given an initial guess $\mathbf{u}^{(0)}$, we obtain each successive iterate as

$$\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + B^{-1}(\mathbf{f} - A\mathbf{u}^{(k)}). \tag{2.2}$$

Here we consider the case when the *iterator* or *preconditioner* B is defined via two-level or two-grid algorithms. Our main interest is convergence of the approximate solutions $\mathbf{u}^{(k)}$ defined via (2.2) to the solution $\mathbf{u}$ of (2.1). This convergence rate is determined by estimating the norm defined by the A-inner product $(.)^TA(.)$ (or the A-norm $\|\cdot\|_A$) of the error transfer operator $E = I - B^{-1}A$. Whenever needed, we will distinguish between two-level hierarchical basis methods and two-grid methods by denoting the corresponding preconditioner B and the error transfer operator E with B_{TL}, E_{TL} and B_{TG}, E_{TG} respectively. The theoretical results and algorithmic constructions we present follow the classical two-level hierarchical basis approach (cf. Bank and Dupont, Braess, Axelsson and Gustafsson, and Yserentant, in the pioneering papers [3, 6, 2, 16, 4]). Most of these methods are summarized in Bank [5] (see also [11] or [14]). As usual, for two-level hierarchical basis methods, we assume that V is decomposed as a *direct sum*

$$V = SV_s + PV_c, \tag{2.3}$$

for some components V_s and V_c isomorphic to $\mathbb{R}^{n_s}$ and $\mathbb{R}^{n_c}$, respectively, with $n = n_s + n_c$. A typical and simple example to keep in mind is $S = \begin{bmatrix} I \\ 0 \end{bmatrix}$ and $P = \begin{bmatrix} W \\ I \end{bmatrix}$ for some W such that the square matrix $[S,\ P]$ is unit upper triangular, and hence invertible. That is, the decomposition (2.3) is direct. For the more general case of "two-grid" methods, one may also assume a decomposition $V = SV_s + PV_c$ as in (2.3), but it does not have to be direct. That is, one may have $n \le n_s + n_c$. Yet another difference is that for two-grid methods only the *coarse* space PV_c is explicitly needed (i.e., a basis is given), whereas the first component (which is not unique) need not actually be specified. Usually, a one to one mapping P defines the coarse space. Such a mapping is commonly referred to as *interpolation* or *prolongation*. One has $PV_c \subset V$, and the coarse space is Range(P). In the two-level hierarchical basis case, the direct decomposition $V = SV_s + PV_c$ is commonly referred to as the two-level hierarchical basis decomposition. The space V_c defines the so–called coarse degrees of freedom (or coarse–grid variables) which are very often in practice obtained from V based on a simple (restriction) mapping R. For the example above one can let $R = [0,\ I]$. Then $V_c = RV$. It is clear that the pair $[S,\ R^T]$ also provides a direct (actually orthogonal) decomposition of V, but the one based on a P with $W \ne 0$ is the typical choice for many practical examples since the respective two–level (and two–grid) methods are likely to perform much better with P than with R^T.

2.1. Smoother, coarse grid matrix and projections

A two-level or two-grid method can be defined if two ingredients are in place. One of them is the space decomposition and the other is the *smoother*. A smoother here will be denoted with M, where the smoother iteration is as in (2.2) with M replacing B. The following result is well known (and easily seen):

$$M^T + M - A \text{ is s.p.d.} \quad \Longleftrightarrow \quad \|I - M^{-1}A\|_A < 1. \tag{2.4}$$

Hence, throughout the remainder of the paper, we will assume that $M^T + M - A$ is s.p.d., or equivalently, that the smoother iteration is a contraction in A-norm.

Various restrictions of M and A to the subspaces mentioned before will be needed. We first define the exact coarse grid matrix A_c and its hierarchical complement A_s as follows

$$A_c = P^T AP, \qquad A_s = S^T AS.$$

Later we will see, in the case of a two level hierarchical preconditioner, one needs M to be well-defined only on the first (hierarchical) component SV_s. In that case, we refer to M as M_s. However, we can think of M_s as being derived from a global (not necessarily symmetric) smoother M, i.e., that $M_s = S^T MS$ where $M^T + M - A$ is positive semi-definite. As an example, consider again the simple case where $S^T = [\ I\ \ 0\ ]$ and let $R = [\ 0\ \ I\ ]$. Then, A admits the following two–by–two block form,

$$A = \begin{bmatrix} S^T \\ R \end{bmatrix} A[S, R^T] = \begin{bmatrix} A_s & S^T AR^T \\ RAS & RAR^T \end{bmatrix}.$$

Then, for a given M_s such that $M_s^T + M_s - A_s$ is positive definite, of interest is the block-factored smoother

$$M = \begin{bmatrix} M_s & 0 \\ RAS & \tau I \end{bmatrix} \begin{bmatrix} I & M_s^{-1}\, S^T AR^T \\ 0 & I \end{bmatrix},$$

where $\tau > 0$ is a sufficiently large constant. Note that (since $A^T = A$)

$$M^T + M - A = \begin{bmatrix} M_s^T + M_s - A_s & S^T A R^T \\ RAS & 2\tau I - RAR^T + (RAS)\left(M_s^{-1} + M_s^{-T}\right)(S^T A R^T) \end{bmatrix},$$

which can be made positive definite if τ is sufficiently large.

The following two operators are related to the smoother M, and will be frequently used in the definitions and analysis later on:

$$\widetilde{M} = M^T(M^T + M - A)^{-1}M, \qquad \overline{M} = M(M^T + M - A)^{-1}M^T. \tag{2.5}$$

Likewise, the operators $\widetilde{M}_s$ and $\overline{M}_s$ are defined by replacing M and A in (2.5) by M_s and A_s. Note that $(I - \widetilde{M}^{-1}A) = (I - M^{-1}A)(I - M^{-T}A)$, hence $\widetilde{M}$ is just a symmetrized version of the smoother M (and similarly for $\overline{M}$). Also note that $\widetilde{M} = \overline{M}$ when M is symmetric, but in general both operators are needed: $\widetilde{M}$ is needed for the error analysis and $\overline{M}$ is needed in the definition of the preconditioners. Finally, we remark that if $M^T + M - A$ is positive definite, then $\widetilde{M} - A$ and $\overline{M} - A$ are positive semidefinite. This is easily seen from the simple relation,

$$\widetilde{M} - A = (X - M)X^{-1}(X - M^T), \text{ with } X = M^T + M - A, \tag{2.6}$$

which, with obvious change, holds for $\overline{M}$ as well.

In what follows we will need two projection operators related to the coarse space Range(P). We define

$$\pi_A = P A_c^{-1} P^T A, \qquad \overline{\pi}_A = A^{\frac{1}{2}} P A_c^{-1} P^T A^{\frac{1}{2}}, \tag{2.7}$$

and observe that π_A is an A-orthogonal projection on Range(P) and $\overline{\pi}_A$ is a $\langle \cdot, \cdot \rangle$-orthogonal projection on Range($A^{\frac{1}{2}}P$).

2.2. The strengthened Cauchy-Schwarz inequality and the Schur complement

In the analysis of the two-level hierarchical preconditioner, we will need the strengthened Cauchy-Schwarz inequality (sometimes called the Cauchy-Bunyakowski-Schwarz, or C.B.S. inequality), which provides a bound on the cosine of the abstract angle between two subspaces. Assume that $V = SV_s + PV_c$ is a direct decomposition and let $\gamma^2 \in [0, 1)$ be the smallest constant in the following inequality

$$(\mathbf{w}^T S^T A P \mathbf{x})^2 \le \gamma^2\, \mathbf{w}^T S^T A S \mathbf{w}\, \mathbf{x}^T P^T A P \mathbf{x}. \tag{2.8}$$

This C.B.S. inequality implies,

$$\mathbf{w}^T A_s \mathbf{w} \le \frac{1}{1-\gamma^2} \inf_{\mathbf{x}} (S\mathbf{w} + P\mathbf{x})^T A (S\mathbf{w} + P\mathbf{x}), \quad \forall \mathbf{w} \in V_s, \quad \forall \mathbf{x} \in V_c.$$

The latter minimum is attained at $P\mathbf{x} = -\pi_A S \mathbf{w}$. Therefore,

$$\begin{aligned} \mathbf{w}^T A_s \mathbf{w} &\le \tfrac{1}{1-\gamma^2}\, \mathbf{w}^T \left(S^T (I - \pi_A)^T A (I - \pi_A) S\right) \mathbf{w} \\ &= \tfrac{1}{1-\gamma^2}\, \mathbf{w}^T \left(S^T A (I - \pi_A) S\right) \mathbf{w}. \end{aligned} \tag{2.9}$$

As it is well known, the constant in the strengthened C.B.S. inequality is related to the spectral equivalence between the Schur complement $\mathcal{S}_A$ of A and $A_c = P^T A P$. The Schur complement $\mathcal{S}_A$ is defined here as

$$\mathbf{x}^T \mathcal{S}_A \mathbf{x} = \inf_{\mathbf{w}} (S\mathbf{w} + P\mathbf{x})^T A (S\mathbf{w} + P\mathbf{x}). \tag{2.10}$$

One has, for any real t, assuming (2.8),

$$\begin{aligned}\mathbf{x}^T \mathcal{S}_A \mathbf{x} &= \inf_{\mathbf{w}} \inf_{t} \,(tS\mathbf{w} + P\mathbf{x})^T A (tS\mathbf{w} + P\mathbf{x}) \\ &= \inf_{\mathbf{w}} \left(\mathbf{x}^T A_c \mathbf{x} - \frac{\left(\mathbf{w}^T S^T A P \mathbf{x}\right)^2}{\mathbf{w}^T A_s \mathbf{w}} \right) \\ &\geq \inf_{\mathbf{w}} \left(\mathbf{x}^T A_c \mathbf{x} - \gamma^2 \, \mathbf{x}^T A_c \mathbf{x}\right) \\ &= (1-\gamma^2)\, \mathbf{x}^T A_c \mathbf{x}.\end{aligned}$$

We will show below that the converse statement is also true.

Lemma 2.1. *Consider the Schur complement $\mathcal{S}_A$ defined in* (2.10). *An equivalent formulation of the C.B.S. inequality* (2.8) *reads then as follows:*

There exists a $\gamma \in [0, 1)$ such that for any $\mathbf{x} \in \mathbb{R}^{n_c}$, one has

$$(1-\gamma^2)\, \mathbf{x}^T A_c \mathbf{x} \leq \mathbf{x}^T \mathcal{S}_A \mathbf{x} \leq \mathbf{x}^T A_c \mathbf{x}. \tag{2.11}$$

Proof. The left hand side of (2.11) together with the definition of the Schur complement $\mathcal{S}_A$, imply $(1-\gamma^2)\, \mathbf{x}^T A_c \mathbf{x} \leq (P\mathbf{x} + tS\mathbf{w})^T A\,(P\mathbf{x} + tS\mathbf{w})$, for any real t. The latter shows

$$0 \leq t^2 \mathbf{w}^T A_s \mathbf{w} + 2t \; \mathbf{w}^T S^T A P \mathbf{x} + \gamma^2 \; \mathbf{x}^T A_c \mathbf{x}.$$

Therefore, the discriminant of the above non–negative quadratic form must be non–positive, which is in fact the C.B.S. inequality (2.8). □

2.3. Two-level and two-grid preconditioners

Having all components in place, we are now in a position to define the classical two-level hierarchical basis method. This method exploits a direct (hierarchical) space decomposition $V = SV_s + PV_c$. Namely, we decompose $\mathbf{u} \in V$ uniquely as $\mathbf{u} = S\mathbf{u}_s + P\mathbf{u}_c$. The problem (2.1) is then transformed to the equivalent one, with the hierarchical basis matrix $\widehat{A} \equiv [S,\ P]^T A [S,\ P]$:

$$\widehat{A} \begin{bmatrix} \mathbf{u}_s \\ \mathbf{u}_c \end{bmatrix} = [S,\ P]^T \, \mathbf{f}.$$

Note that

$$\widehat{A} = \begin{bmatrix} A_s & S^T A P \\ P^T A S & A_c \end{bmatrix}.$$

This transformed matrix is then used to define the preconditioner B_{TL} in terms of its hierarchical counterpart $\widehat{B}_{TL}$.

Definition 2.1 (Two-level hierarchical basis preconditioner, B_{TL}) *Let*

$$\widehat{B}_{TL} = \begin{bmatrix} I & 0 \\ P^T A S M_s^{-1} & I \end{bmatrix} \begin{bmatrix} \overline{M}_s & 0 \\ 0 & A_c \end{bmatrix} \begin{bmatrix} I & M_s^{-T} S^T A P \\ 0 & I \end{bmatrix}.$$

Then, the two-level hierarchical basis preconditioner B_{TL} is defined by returning to the original variables,

$$B_{TL}^{-1} = [S,\ P] \widehat{B}_{TL}^{-1} [S,\ P]^T.$$

It is clear, that to implement the actions of B_{TL}^{-1}, one needs the actions of the smoothers M_s^{-1}, M_s^{-T}, a coarse-grid solver to evaluate A_c^{-1}, the actions of the interpolation (P) and restriction (P^T) mappings, as well as the ability to extract the hierarchical component $S\mathbf{v}_s$ of any vector $\mathbf{v} = S\mathbf{v}_s + R^T\mathbf{v}_c$, which defines S and similarly for S^T. We comment on the fact that the above definition takes the point of view that B_{TL} is being obtained via a *product iteration scheme*, in the sense that the error transfer operator E_{TL} has the following form

$$E_{TL} \equiv (I - SM_s^{-T}S^TA)(I - PA_c^{-1}P^TA)(I - SM_s^{-1}S^TA).$$

The latter represents a composite subspace iteration process; the first step being smoothing based on M_s in the first coordinate space SV_s, followed by an (exact) coarse-grid correction in the subspace PV_c and finally followed by a post-smoothing based on M_s^T in the first coordinate space SV_s. From the preconditioning point of view though, one can simplify the definition somewhat and thus end up with the following "approximate block-factorization" preconditioner in the case of symmetric M_s such that $M_s - A_s$ is positive semi–definite:

$$\widehat{B}_{TL}^a = \begin{bmatrix} M_s & 0 \\ P^TAS & I \end{bmatrix} \begin{bmatrix} M_s^{-1} & 0 \\ 0 & B_c \end{bmatrix} \begin{bmatrix} M_s & S^TAP \\ 0 & I \end{bmatrix}. \tag{2.12}$$

Then, $(B_{TL}^a)^{-1} = [S,\ P](\widehat{B}_{TL}^a)^{-1}[S,\ P]^T$ is the inverse "approximate block-factorization" preconditioner to A. This was the preconditioner originally studied in Axelsson and Gustafsson [2] with B_c being an s.p.d. approximation to A_c.

An important observation is that in order to define $B_{TL}^{-1} = [S,\ P]\widehat{B}_{TL}^{-1}[S,\ P]^T$, one does not have to assume that $[S,\ P]$ is invertible, or even square. Thus, one can formally let $S = I$, and hence $M_s = M$, and the resulting method gives the classical two-grid preconditioner B_{TG} with the corresponding error transfer operator defined by,

$$E_{TG} \equiv (I - M^{-T}A)(I - PA_c^{-1}P^TA)(I - M^{-1}A). \tag{2.13}$$

A more precise definition is as follows.

Definition 2.2 (Two-grid preconditioner, B_{TG}) *Let*

$$\widehat{B}_{TG} = \begin{bmatrix} I & 0 \\ P^TAM^{-1} & I \end{bmatrix} \begin{bmatrix} \overline{M} & 0 \\ 0 & A_c \end{bmatrix} \begin{bmatrix} I & M^{-T}AP \\ 0 & I \end{bmatrix}. \tag{2.14}$$

Then, the two-grid preconditioner B_{TG} is defined by

$$B_{TG}^{-1} = [I,\ P]\,\widehat{B}_{TG}^{-1}\,[I,\ P]^T. \tag{2.15}$$

Note that $\widehat{B}_{TG} :\ \mathbb{R}^{n+n_c} \mapsto \mathbb{R}^{n+n_c}$ has a "bigger" size than B_{TG} and A; namely, it defines an operator acting on the product space $V \times \text{Range}(P)$.

Lemma 2.2. *The two–grid preconditioner B_{TG} used in a stationary iterative method gives rise to the iteration matrix E_{TG} defined in* (2.13)*, i.e.,*

$$I - B_{TG}^{-1}A = E_{TG}.$$

Proof. From (2.14), a straightforward calculation of the inverse $\widehat{B}_{TG}^{-1}$, gives

$$\widehat{B}_{TG}^{-1} = \begin{bmatrix} I & -M^{-T}AP \\ 0 & I \end{bmatrix} \begin{bmatrix} \overline{M}^{-1} & 0 \\ 0 & A_c^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -P^TAM^{-1} & I \end{bmatrix}. \tag{2.16}$$

Then, forming the right-hand side of (2.15) leads to

$$\begin{aligned} B_{TG}^{-1} &= \left[I,\ \left(I - M^{-T}A\right)P\right] \left[\begin{array}{cc} \overline{M}^{-1} & 0 \\ 0 & A_c^{-1} \end{array}\right] \left[\begin{array}{c} I \\ P^T\left(I - AM^{-1}\right) \end{array}\right] \\ &= \overline{M}^{-1} + \left(I - M^{-T}A\right)PA_c^{-1}P^T\left(I - AM^{-1}\right). \end{aligned}$$

Finally, since $I - \overline{M}^{-1}A = (I - M^{-T}A)(I - M^{-1}A)$, one arrives at,

$$\begin{aligned} I - B_{TG}^{-1}A &= I - \overline{M}^{-1}A - \left(I - M^{-T}A\right)PA_c^{-1}P^T\left(I - AM^{-1}\right)A \\ &= (I - M^{-T}A)(I - M^{-1}A) - \left(I - M^{-T}A\right)PA_c^{-1}P^TA\left(I - M^{-1}A\right) \\ &= (I - M^{-T}A)(I - PA_c^{-1}P^TA)\left(I - M^{-1}A\right) \\ &= E_{TG}. \end{aligned}$$

□

3. A saddle-point lemma

A crucial identity that will be used to derive the spectral equivalence results is the following lemma, henceforth referred to as the "saddle-point lemma". We state the lemma in a somewhat abstract form for two vector spaces V_1 and V_2. We will use it with $V_1 = V_s$ for the two-level hierarchical basis preconditioner, and with $V_1 = V$ for the two grid preconditioner. The second space V_2 will be taken to be the range of a projection on a subspace of V.

Lemma 3.1. *Given two mappings $T\ : V_1 \mapsto V_1$ and $N :\ V_1 \mapsto V_2$ such that $T + N^TN$ is invertible, with T symmetric positive semi-definite and N onto (i.e., for any vector $\mathbf{v} \in V_2$ the equation $N\mathbf{w} = \mathbf{v}$ has at least one solution $\mathbf{w} \in V_1$), consider the mapping $Z = N\left(T + N^TN\right)^{-1}N^T$. We have that Z is s.p.d., and the following identity holds:*

$$\frac{\mathbf{v}^TZ^{-1}\mathbf{v}}{\mathbf{v}^T\mathbf{v}} = 1 + \inf_{\mathbf{w}:\ N\mathbf{w}=\mathbf{v}} \frac{\mathbf{w}^TT\mathbf{w}}{\mathbf{v}^T\mathbf{v}}. \tag{3.1}$$

Proof. We first remark that N being onto is equivalent to N^T having full column rank. It is also easy to see that T is (symmetric) positive definite on the null-space of N.

Consider now the following quadratic constrained minimization problem: Given $\mathbf{v} \in V_2$, find a $\mathbf{w} \in V_1$ that solves

$$\begin{array}{c} \frac{1}{2}\mathbf{w}^TT\mathbf{w} \mapsto \min \\ \text{subject to } N\mathbf{w} = \mathbf{v}. \end{array} \tag{3.2}$$

By forming the Lagrangian $\mathcal{L}(\mathbf{w},\ \lambda) = \frac{1}{2}\mathbf{w}^TT\mathbf{w} + \lambda^T(N\mathbf{w} - \mathbf{v})$ and setting its partial derivatives to zero we get the following saddle-point problem for $\mathbf{w}$ and the Lagrange multiplier λ,

$$\left[\begin{array}{cc} T & N^T \\ N & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{w} \\ \lambda \end{array}\right] = \left[\begin{array}{c} 0 \\ \mathbf{v} \end{array}\right].$$

Our assumptions on T and N (namely, T being positive definite on the null-space of N and N^T having full column rank) tell us that the above problem has a unique solution $(\mathbf{w}_*,\ \lambda_*)$. It is also clear that the (negative) Schur complement $Z = N\left(T + N^TN\right)^{-1} N^T$ of the saddle-point matrix

$$\begin{bmatrix} T + N^TN & N^T \\ N & 0 \end{bmatrix},$$

is. s.p.d. (hence invertible). Since $(\mathbf{w}_*,\ \lambda_*)$ solves the equivalent problem,

$$\begin{bmatrix} T + N^TN & N^T \\ N & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \lambda \end{bmatrix} = \begin{bmatrix} N^T\mathbf{v} \\ \mathbf{v} \end{bmatrix},$$

one gets the identity,

$$\mathbf{w}_* = \left(T + N^TN\right)^{-1} N^T(\mathbf{v} - \lambda_*),$$

which implies

$$\mathbf{v} = N\mathbf{w}_* = N\left(T + N^TN\right)^{-1} N^T(\mathbf{v} - \lambda_*) = Z(\mathbf{v} - \lambda_*).$$

Hence, $\mathbf{v} - \lambda_* = Z^{-1}\mathbf{v}$, and therefore $\mathbf{v}^T\mathbf{v} - \mathbf{v}^T\lambda_* = \mathbf{v}^TZ^{-1}\mathbf{v}$. The latter implies (using $\mathbf{v} = N\mathbf{w}_*$, and $N^T\lambda_* = -T\mathbf{w}_*$)

$$\mathbf{v}^T\mathbf{v} + \mathbf{w}_*^TT\mathbf{w}_* = \mathbf{v}^TZ^{-1}\mathbf{v}.$$

Next, since $\mathbf{w}_*$ solves the constrained minimization problem (3.2) we arrive at the desired identity (3.1). □

4. Sharp spectral equivalence results

Our main goal in this section will be to obtain a suitable expression for the best constant K taking part in the spectral equivalence relations between A and B, for $B = B_{TL}$ and $B = B_{TG}$:

$$\mathbf{v}^TA\mathbf{v} \le \mathbf{v}^TB\mathbf{v} \le K\ \mathbf{v}^TA\mathbf{v}. \tag{4.1}$$

We will try to handle both cases (the hierarchical two-level and the two-grid one) simultaneously by introducing the notation: $\mathcal{M} \equiv J^TMJ$ and $\mathcal{A} \equiv J^TAJ$, where either $J = S$ or $J = I$. That is, either $\mathcal{M} = S^TMS = M_s$ and $\mathcal{A} = S^TAS = A_s$, or $\mathcal{M} = M$ and $\mathcal{A} = A$. With this, and considering also the case of $B = \widehat{B}^a_{TL}$, we can write the general error transfer operator for (2.2) as follows (the specific operators E_{TL} and E_{TG} were defined in Section 2.3):

$$I - B^{-1}A = E \equiv (I - J\mathcal{M}^{-T}J^TA)(I - P\mathcal{D}^{-1}P^TA)(I - J\mathcal{M}^{-1}J^TA), \tag{4.2}$$

where $\mathcal{D}$ is an s.p.d. approximation to the coarse-grid matrix $A_c = P^TAP$. Of particular importance is the case $\mathcal{D} = A_c$, which we consider in great detail, but we also point out how the convergence rate can be estimated using an appropriate approximation $\mathcal{D}$ to A_c.

Multiplying both sides of (4.2) by A we get that

$$AE = A^{\frac{1}{2}}(I - A^{\frac{1}{2}}J\mathcal{M}^{-T}J^TA^{\frac{1}{2}})(I - A^{\frac{1}{2}}P\mathcal{D}^{-1}P^TA^{\frac{1}{2}})(I - A^{\frac{1}{2}}J\mathcal{M}^{-1}J^TA^{\frac{1}{2}})A^{\frac{1}{2}}. \tag{4.3}$$

Let us denote for a moment $X = (I - A^{\frac{1}{2}}P\mathcal{D}^{-1}P^TA^{\frac{1}{2}})$. Note that if $\mathcal{D}$ is defined such that $\mathbf{v}_c^T\mathcal{D}\mathbf{v}_c \ge \mathbf{v}_c^TA_c\mathbf{v}_c$, then both X (see Lemma 4.1 for details) and AE are symmetric positive semi-definite. Equivalently, one has that $A^{\frac{1}{2}}EA^{-\frac{1}{2}} = I - A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$ is also symmetric positive semi–definite, which guarantees that the left-hand inequality in (4.1) holds assuming that B^{-1} is s.p.d. The latter follows from the fact that $\|E\|_A \leq 1$, which is seen from (4.2) and (2.4).

Consider now the right-hand inequality in (4.1). Since AE is symmetric positive semi-definite, then $\|E\|_A$ is given by the largest eigenvalue of $A^{\frac{1}{2}}EA^{-\frac{1}{2}}$, or equivalently by $1 - \frac{1}{K}$. But, X being symmetric positive semi–definite implies that $X^{\frac{1}{2}}$ is well-defined, and it is obvious from $\|G\| = \|G^T\|$, for $G = X^{\frac{1}{2}}(I - A^{\frac{1}{2}}J\mathcal{M}^{-1}J^TA^{\frac{1}{2}}))$, that the largest eigenvalue of

$$A^{\frac{1}{2}}EA^{-\frac{1}{2}} = (I - A^{\frac{1}{2}}J\mathcal{M}^{-T}J^TA^{\frac{1}{2}})X(I - A^{\frac{1}{2}}J\mathcal{M}^{-1}J^TA^{\frac{1}{2}})$$

is the same as the the largest eigenvalue of

$$\Theta \equiv X^{\frac{1}{2}}(I - A^{\frac{1}{2}}J\mathcal{M}^{-1}J^TA^{\frac{1}{2}})(I - A^{\frac{1}{2}}J\mathcal{M}^{-T}J^TA^{\frac{1}{2}})X^{\frac{1}{2}}.$$

Therefore, we will proceed with estimating the last expression.

In what follows we consider the case $\mathcal{D} = A_c$. The inexact coarse matrix $\mathcal{D}$ will be commented on in more detail in Section 4.1. One notices then that $X = I - \overline{\pi}_A$ and the square root can be removed, because $X^2 = X$ (and hence $X = X^{\frac{1}{2}}$) in this case. Recalling the definition (2.5) of $\widetilde{M}$, consider then the expression

$$\begin{aligned}
\mathbf{v}^T\Theta\mathbf{v} &= (X\mathbf{v})^T\left(I - A^{\frac{1}{2}}J\left(\mathcal{M}^{-1} + \mathcal{M}^{-T} - \mathcal{M}^{-1}(J^TAJ)\mathcal{M}^{-T}\right)J^TA^{\frac{1}{2}}\right)(X\mathbf{v})\\
&= (X\mathbf{v})^T\left(I - A^{\frac{1}{2}}J\left(\mathcal{M}^{-1} + \mathcal{M}^{-T} - \mathcal{M}^{-1}\mathcal{A}\mathcal{M}^{-T}\right)\ J^TA^{\frac{1}{2}}\right)(X\mathbf{v})\\
&= (X\mathbf{v})^T\left(I - A^{\frac{1}{2}}J\widetilde{\mathcal{M}}^{-1}J^TA^{\frac{1}{2}}\right)(X\mathbf{v})\\
&= \mathbf{v}^T(I - \overline{\pi}_A)^2\mathbf{v} - \mathbf{v}^T(I - \overline{\pi}_A)A^{\frac{1}{2}}J\widetilde{\mathcal{M}}^{-1}J^TA^{\frac{1}{2}}(I - \overline{\pi}_A)\mathbf{v}\\
&= ((I - \overline{\pi}_A)\mathbf{v})^T\left(I - (I - \overline{\pi}_A)A^{\frac{1}{2}}J\widetilde{\mathcal{M}}^{-1}J^TA^{\frac{1}{2}}(I - \overline{\pi}_A)\right)\ (I - \overline{\pi}_A)\mathbf{v}\\
&\leq \left(1 - \tfrac{1}{K}\right)((I - \overline{\pi}_A)\mathbf{v})^T\ (I - \overline{\pi}_A)\mathbf{v}.
\end{aligned}$$

The smallest (i.e. the best) constant K in the above inequality can then be defined via the following relation

$$\frac{1}{K} = \inf_{\mathbf{v}} \frac{\mathbf{v}^T\left((I - \overline{\pi}_A)A^{\frac{1}{2}}J\widetilde{\mathcal{M}}^{-1}J^TA^{\frac{1}{2}}(I - \overline{\pi}_A)\right)\mathbf{v}}{\mathbf{v}^T(I - \overline{\pi}_A)\mathbf{v}}. \tag{4.4}$$

Since above in (4.4) we have $\widetilde{\mathcal{M}}^{-1}$, we will use Lemma 3.1 to get an estimate in terms of $\widetilde{\mathcal{M}}$ instead in the next theorem. We would also like to point out that the expression for K in (4.5) below is related to the identity of Xu and Zikatanov [15] (valid for abstract iteration methods). Although it may be possible, the derivation of the relation (4.5) with the techniques from [15] is not at all straightforward, and our direct proof here is better suited for the convergence analysis of two-level and two-grid methods given later on.

Theorem 4.1. *Assume that J and P are such that any vector $\mathbf{v}$ can be decomposed as $\mathbf{v} = J\mathbf{w} + P\mathbf{x}$. Then the best constant K in (4.1) is given by*

$$K = \sup_{\mathbf{v}\in Range(I-\pi_A)}\ \inf_{\mathbf{w}:\ \mathbf{v}=(I-\pi_A)J\mathbf{w}} \frac{\mathbf{w}^T\widetilde{\mathcal{M}}\mathbf{w}}{\mathbf{v}^TA\mathbf{v}}. \tag{4.5}$$

Proof. Let $N = (I - \overline{\pi}_A)A^{\frac{1}{2}}J$. Then

$$N^T N = J^T \left(A^{\frac{1}{2}}(I - \overline{\pi}_A)^2 A^{\frac{1}{2}}\right) J = J^T A(I - \pi_A)J.$$

Define

$$T = \widetilde{\mathcal{M}} - J^T A(I - \pi_A)J = \widetilde{\mathcal{M}} - \mathcal{A} + J^T A\pi_A J.$$

It is clear that T is symmetric positive semi-definite. We also have,

$$\widetilde{\mathcal{M}} = T + N^T N.$$

We point out here, that in order to apply Lemma 3.1, a minor (but very important) detail needs to be checked out; namely, whether or not the mapping $N : V_s \mapsto \text{Range}(I - \overline{\pi}_A)$, when $J = S$ is onto. This is done in the following way. Let $\mathbf{x} \neq 0$, $\mathbf{x} \in \text{Range}(I - \overline{\pi}_A)$ be given. We will find a $\mathbf{w} \in V_s$, such that $N\mathbf{w} = \mathbf{x}$. Since $\mathbf{x} \in \text{Range}(I - \overline{\pi}_A)$ there is a $\mathbf{v} \in V$, such that $\mathbf{x} = (I - \overline{\pi}_A)\mathbf{v}$. By the assumptions of the theorem, the following decompositions hold,

$$A^{-1/2}\mathbf{v} = J\mathbf{w} + P\mathbf{y}, \quad \text{and} \;\; \mathbf{v} = A^{1/2}J\mathbf{w} + A^{1/2}P\mathbf{y}.$$

Since $(I - \overline{\pi}_A)A^{\frac{1}{2}}P = 0$, one gets that $\mathbf{x} = N\mathbf{w}$. Then the "saddle-point" lemma 3.1 with $Z = N\left(T + N^T N\right)^{-1} N^T = (I - \overline{\pi}_A)A^{\frac{1}{2}}J\widetilde{\mathcal{M}}^{-1}J^T A^{\frac{1}{2}}(I - \overline{\pi}_A)$ and $V_2 = \text{Range}(I - \overline{\pi}_A)$, applied to identity (4.4), gives us the new identity

$$\frac{1}{K} = \inf_{\mathbf{v}\in V_2} \frac{\mathbf{v}^T Z\mathbf{v}}{\mathbf{v}^T\mathbf{v}} = \inf_{\mathbf{v}\in V_2} \frac{\mathbf{v}^T\mathbf{v}}{\mathbf{v}^T Z^{-1}\mathbf{v}} = \frac{1}{1 + \sup\limits_{\mathbf{v}\in V_2} \inf\limits_{\mathbf{w}:\, N\mathbf{w}=\mathbf{v}} \frac{\mathbf{w}^T T\mathbf{w}}{\mathbf{v}^T\mathbf{v}}}. \tag{4.6}$$

Further, in the denominator on the right hand side of (4.6) we have $\mathbf{v} = N\mathbf{w} = (I - \overline{\pi}_A)A^{\frac{1}{2}}J\mathbf{w}$. Replace now $\mathbf{v} \;\equiv A^{\frac{1}{2}}\mathbf{v}$. This implies that $N\mathbf{w} = A^{\frac{1}{2}}\mathbf{v}$, or $A^{\frac{1}{2}}\mathbf{v} = (I - \overline{\pi}_A)A^{\frac{1}{2}}J\mathbf{w}$; that is, $\mathbf{v} = (I - \pi_A)J\mathbf{w}$. Identity (4.6) then leads to (noticing that now $\mathbf{v} \in \text{Range}(I - \pi_A)$)

$$K = 1 + \sup_{\mathbf{v}\in\text{Range}(I-\pi_A)} \inf_{\mathbf{w}:\, \mathbf{v}=(I-\pi_A)J\mathbf{w}} \frac{\mathbf{w}^T T\mathbf{w}}{\mathbf{v}^T A\mathbf{v}}.$$

We also have,

$$\mathbf{w}^T T\mathbf{w} = \mathbf{w}^T\widetilde{\mathcal{M}}\mathbf{w} - \mathbf{w}^T N^T N\mathbf{w} = \mathbf{w}^T\widetilde{\mathcal{M}}\mathbf{w} - \mathbf{v}^T A\mathbf{v}.$$

Substituting the last expression in the above formula for K, implies the desired result (4.5). □

Corollary 4.1. *Assume that S and P provide a unique decomposition; namely, that $[S,\, P]$ is an invertible square matrix. Then,*

$$K = \sup_{\mathbf{w}} \frac{\mathbf{w}^T\widetilde{M}_s\mathbf{w}}{\mathbf{w}^T S^T(I - \pi_A)A(I - \pi_A)S\mathbf{w}}. \tag{4.7}$$

Proof. Note that for $\mathbf{v} = (I - \pi_A)S\mathbf{w}$, we also have $\mathbf{v} = S\mathbf{w} + P(-A_c^{-1}P^T A)(S\mathbf{w})$. Then, since S and P provide a unique decomposition of $\mathbf{v} = S\mathbf{w} + P\mathbf{x}$, this shows that the second component of $\mathbf{v}$ (which is in fact unique) equals $\mathbf{x} = -A_c^{-1}(P^T A)(S\mathbf{w})$. I.e., there is no inf in the formula for K. Thus,

$$K = \sup_{\mathbf{w}} \frac{\mathbf{w}^T\widetilde{M}_s\mathbf{w}}{\mathbf{w}^T S^T A(I - \pi_A)S\mathbf{w}},$$

which is the same as (4.7). □

4.1. Analysis of the two-level hierarchical basis preconditioner B_{TL}

We will now derive an upper bound for K in the case of S (i.e., $J = S$) and P providing a unique decomposition. We recall that $\mathcal{A} = S^T A S = A_s$. The following lemma gives the monotone dependence of $B - A$ on $\mathcal{D} - A_c$, and is needed in proving Theorem 4.2. We recall that $\mathcal{D}$ is a spectrally equivalent approximation to the exact coarse matrix $A_c = P^T A P$ (used in (4.2)).

Lemma 4.1. *If $\mathcal{D} - A_c$ is symmetric positive semidefinite, then $B - A$ is symmetric positive semidefinite.*

Proof. Since $A_c = P^T A P$, and hence $\|G\| = \|G^T\| = 1$ for $G = A_c^{-\frac{1}{2}} P^T A^{\frac{1}{2}}$, we have that

$$\mathbf{v}^T A^{-1} \mathbf{v} \geq (P^T \mathbf{v})^T A_c^{-1} (P^T \mathbf{v}).$$

This and the assumptions of the lemma lead to the inequality

$$\mathbf{v}^T A^{-1} \mathbf{v} \geq (P^T \mathbf{v})^T \mathcal{D}^{-1} (P^T \mathbf{v}),$$

which is equivalent to

$$\mathbf{v}^T \left(I - A^{\frac{1}{2}} P \mathcal{D}^{-1} P^T A^{\frac{1}{2}} \right) \mathbf{v} \geq 0.$$

Hence, $AE = A(A^{-1} - B^{-1})A$ is symmetric positive semi-definite, which is equivalent to $B - A$ being symmetric positive semi-definite. □

Theorem 4.2. *Assume that M_s provides a convergent splitting for A_s in the A_s-inner product, i.e., that $\left(M_s + M_s^T - A_s\right)$ is s.p.d. Also, let $\gamma \in [0, 1)$ be the constant in the strengthened Cauchy-Schwarz inequality* (2.8)*. Then B_{TL} (i.e., B with $\mathcal{D} = A_c$; see Definition 2.1) and A are spectrally equivalent and the following bounds hold:*

$$\mathbf{v}^T A \mathbf{v} \leq \mathbf{v}^T B_{TL} \mathbf{v} \leq K \mathbf{v}^T A \mathbf{v}, \quad K \leq \frac{1}{1 - \gamma^2} \sup_{\mathbf{w}} \frac{\mathbf{w}^T \widetilde{M}_s \mathbf{w}}{\mathbf{w}^T A_s \mathbf{w}}. \tag{4.8}$$

In the case of B with an inexact second block $\mathcal{D}$ that satisfies

$$0 \leq \mathbf{x}^T (\mathcal{D} - A_c) \mathbf{x} \leq \Delta \ \mathbf{x}^T A_c \mathbf{x},$$

the following perturbation result holds,

$$\mathbf{v}^T A \mathbf{v} \leq \mathbf{v}^T B \mathbf{v} \leq \left(K + \frac{\Delta}{1 - \gamma^2} \right) \mathbf{v}^T A \mathbf{v}.$$

Proof. The estimate (4.8) follows from (4.7) and (2.9), which combined give the following upper bound for K,

$$K \leq \sup_{\mathbf{w}} \frac{\mathbf{w}^T \widetilde{M}_s \mathbf{w}}{\mathbf{w}^T A_s \mathbf{w}} \ \frac{1}{1 - \gamma^2},$$

The proof is completed by using Lemma 4.1, (2.11), and some obvious inequalities as follows:

$$\begin{aligned} 0 \leq \mathbf{v}^T (B - A) \mathbf{v} \ &= \mathbf{v}^T (B_{TL} - A) \mathbf{v} + \mathbf{x}^T (\mathcal{D} - A_c) \mathbf{x}, \quad \mathbf{x} = P^T \mathbf{v}, \\ &\leq \mathbf{v}^T (B_{TL} - A) \mathbf{v} + \Delta \mathbf{x}^T A_c \mathbf{x} \\ &\leq \mathbf{v}^T (B_{TL} - A) \mathbf{v} + \tfrac{\Delta}{1 - \gamma^2} \ \mathbf{x}^T \mathcal{S}_A \mathbf{x} \\ &\leq \mathbf{v}^T (B_{TL} - A) \mathbf{v} + \tfrac{\Delta}{1 - \gamma^2} \ \mathbf{v}^T A \mathbf{v} \\ &\leq \left(K - 1 + \tfrac{\Delta}{1 - \gamma^2} \right) \mathbf{v}^T A \mathbf{v}. \end{aligned}$$

□

The latter result is a generalization of a main two-level convergence theorem found in [2]. A more detailed study of the (two-level) approximate block-factorization preconditioners B^a_{TL} (defined as in (2.12)) is found in Chapter 9 of [1]. Note that in the classical two-level hierarchical basis methods, A_s happens to be well-conditioned, so it is not that impractical to let $M_s = A_s$. Then, $K = \frac{1}{1-\gamma^2}$ (if $\mathcal{D} = A_c$), i.e., the two-level convergence factor $\varrho(E_{TL}) = 1 - \frac{1}{K} = \gamma^2$ equals to the $\cos^2$ of the abstract angle between the hierarchical components Range(S) and Range(P), a well-known classical result.

4.2. Analysis of the two-grid preconditioner B_{TG}

Here, we consider the case of $B = B_{TG}$, i.e., $J = I$. The analysis follows the lines of the analysis from the previous Section 4.1, and in a similar fashion one obtains that the smallest constant K is given by the identity (see (4.5) in Theorem 4.1)

$$K = \sup_{\mathbf{v}\in \text{Range}(I-\pi_A)} \inf_{\mathbf{w}:\ \mathbf{v}=(I-\pi_A)\mathbf{w}} \frac{\mathbf{w}^T\widetilde{M}\mathbf{w}}{\mathbf{v}^TA\mathbf{v}}.$$

That is,

$$K = \sup_{\mathbf{v}} \frac{\inf_{\mathbf{w}} (\pi_A\mathbf{w} + (I-\pi_A)\mathbf{v})^T\, \widetilde{M}\, (\pi_A\mathbf{w} + (I-\pi_A)\mathbf{v})}{((I-\pi_A)\mathbf{v})^T\, A\, ((I-\pi_A)\mathbf{v})}.$$

The inf over $\mathbf{w}$ is attained at $\mathbf{w}:\ \pi_A(\mathbf{v}-\mathbf{w}) = \pi_{\widetilde{M}}\mathbf{v}$, that is,

$$A_c^{-1}P^TA(\mathbf{v}-\mathbf{w}) = \widetilde{M}_c^{-1}P^T\widetilde{M}\mathbf{v}.$$

Here $\widetilde{M}_c = P^T\widetilde{M}P$ and $\pi_{\widetilde{M}} = P\widetilde{M}_c^{-1}P^T\widetilde{M}$. For a $\mathbf{w} = P\mathbf{w}_c$ one has $A_c^{-1}P^TA\mathbf{w} = \mathbf{w}_c$. Hence,

$$\mathbf{w}_c = A_c^{-1}P^TA\mathbf{v} - \widetilde{M}_c^{-1}P^T\widetilde{M}\ \mathbf{v}.$$

This implies, $\pi_A\mathbf{w} = P\mathbf{w}_c = (\pi_A - \pi_{\widetilde{M}})\mathbf{v}$. Therefore, $\pi_A\mathbf{w} + (I-\pi_A)\mathbf{v} = (I-\pi_{\widetilde{M}})\mathbf{v}$. Thus we arrive at the final estimate which is formulated in the next theorem.

Theorem 4.3. *The convergence factor of the two-grid method,* $\varrho(E_{TG}) = 1 - \frac{1}{K}$, *is characterized by*

$$K = \sup_{\mathbf{v}} \frac{\left((I-\pi_{\widetilde{M}})\mathbf{v}\right)^T \widetilde{M}(I-\pi_{\widetilde{M}})\mathbf{v}}{((I-\pi_A)\mathbf{v})^TA(I-\pi_A)\mathbf{v}} = \sup_{\mathbf{v}} \frac{\mathbf{v}^T\widetilde{M}(I-\pi_{\widetilde{M}})\mathbf{v}}{\mathbf{v}^TA\mathbf{v}}. \quad (4.9)$$

Proof. The second identity in (4.9) is based on the relation $(I-\pi_{\widetilde{M}})(I-\pi_A) = I - \pi_{\widetilde{M}}$ and the inequality $\mathbf{v}^TA(I-\pi_A)\mathbf{v} \le \mathbf{v}^TA\mathbf{v}$. □

Remark 4.1. *Consider the case when the coarse degrees of freedom are defined on the basis of a mapping* $R:\ \mathbb{R}^n \mapsto \mathbb{R}^{n_c}$ *such that* $RP = I$. *Let* $Q = PR$. *Note that* Q *is a projection (i.e.,* $Q^2 = Q$*). Next, noticing that*

$$\|(I-\pi_{\widetilde{M}})\mathbf{e}\|^2_{\widetilde{M}} = \inf_{\mathbf{v}\in Range(P)} \|\mathbf{e}-\mathbf{v}\|^2_{\widetilde{M}} \le \|\mathbf{e} - PR\mathbf{e}\|^2_{\widetilde{M}},$$

one gets the upper bound $K \leq \sup_{\mathbf{e}} \ \mu_{\widetilde{M}}(Q, \ \mathbf{e})$*, where the latter quantity (sometimes called measure) is given by*

$$\mu_{\widetilde{M}}(Q, \ \mathbf{e}) = \frac{\langle \widetilde{M}(I-Q)\mathbf{e}, \ (I-Q)\mathbf{e}\rangle}{\langle A\mathbf{e}, \mathbf{e}\rangle}.$$

The measure $\mu_{\widetilde{M}}(Q, \ \mathbf{e})$ *was the main ingredient used in [12] to find sufficient conditions for the two-grid convergence of the respective AMG method. With the above Theorem 4.3 in hand, we are now able to show that the conditions from [12] are also necessary (in the sense of the next two corollaries 4.2–4.3).*

Corollary 4.2. *For any* $\mathbf{v}$ *in the space Range*$(I - \pi_{\widetilde{M}})$*, one has the spectral equivalence relations,*

$$\mathbf{v}^T A\mathbf{v} \leq \mathbf{v}^T \widetilde{M}\mathbf{v} \leq K \ \mathbf{v}^T A\mathbf{v}, \tag{4.10}$$

where K *is defined via (4.9). Moreover, if* S *is such that Range*(S) = *Range*$(I - \pi_{\widetilde{M}})$*, an analogous spectral equivalence relations between* $A_s = S^T AS$ *and* $\widetilde{M}_s = S^T \widetilde{M} S$ *holds, namely:*

$$\mathbf{v}_s^T A_s \mathbf{v}_s \leq \mathbf{v}_s^T \widetilde{M}_s \mathbf{v}_s \leq K \ \mathbf{v}_s^T A_s \mathbf{v}_s.$$

Proof. Using the inequalities

$$\|(I - \pi_A)\mathbf{w}\|_A = \inf_{\mathbf{v} \in \mathrm{Range}(P)} \ \|\mathbf{w} - \mathbf{v}\|_A \leq \|(I - \pi_{\widetilde{M}})\mathbf{w}\|_A,$$

and $\mathbf{v}^T A\mathbf{v} \leq \mathbf{v}^T \widetilde{M}\mathbf{v}$, the result is straightforward. □

Evidently, one may restate (4.10) by saying that in a space complementary to the coarse space Range(P), the symmetrized smoother $\widetilde{M}$ is an efficient preconditioner for A. Letting $R_* = \widetilde{M}_c^{-1} P^T \widetilde{M}$, one candidate for an S that spans the space Range$(I - \pi_{\widetilde{M}})$ is $S = I - PR_*$. One notices that $R_* P = I$ (and $R_* S = 0$), i.e., we are in the setting of Remark 4.1 with $R = R_*$. We point out that R_* is the optimal mapping that defines the coarse degrees of freedom, in the sense that such a choice will provide the best convergence rate. Note that, for given sparse operators P and M (or its symmetrized version $\widetilde{M}$), the optimal R is not in general sparse, due to presence of $\widetilde{M}_c^{-1}$ in its definition. However, in the following important example, R_* happens to be the simple injection mapping.

Example 4.1. *Consider a two–by–two block partitioning of* A*,*

$$A = \begin{bmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{bmatrix},$$

corresponding to a "f" and "c" block–partitioning of the vectors $\mathbf{v} = \begin{bmatrix} \mathbf{v}_f \\ \mathbf{v}_c \end{bmatrix}$ *in* V*. Introduce the splitting* $A = D_A - L - U$*, with*

$$D_A = \begin{bmatrix} A_{ff} & 0 \\ 0 & A_{cc} \end{bmatrix}, \ -L = \begin{bmatrix} 0 & 0 \\ A_{cf} & 0 \end{bmatrix}, \ \textit{and } U = L^T.$$

For two given (not necessarily symmetric) matrices M_{ff} *and* M_{cc}*, consider the interpolation matrix*

$$P = \begin{bmatrix} -M_{ff}^{-1} A_{fc} \\ I \end{bmatrix},$$

and the (inexact) block Gauss–Seidel smoother (also called "c"–"f" relaxation)

$$M = \begin{bmatrix} M_{ff} & A_{fc} \\ 0 & M_{cc} \end{bmatrix} = D - U,\ D = \begin{bmatrix} M_{ff} & 0 \\ 0 & M_{cc} \end{bmatrix}$$

Then, $R_* = \widetilde{M}_c^{-1} P^T \widetilde{M} = [0,\ I]$*, i.e.,* R_* *is the trivial injection mapping. The optimal* S *defined as* $S_* = I - \pi_{\widetilde{M}}$ *takes the simple form* $S_* = I - PR_* = \begin{bmatrix} I & M_{ff}^{-1}A_{fc} \\ 0 & 0 \end{bmatrix}$*. The latter shows that* $Range(S_*) = Range \begin{bmatrix} I \\ 0 \end{bmatrix}$*. Finally, the corresponding two–grid convergence factor* $\varrho(E_{TG}) = 1 - \frac{1}{K}$ *is characterized with the identity (assuming that* $\|I - M^{-1}A\|_A < 1$*),*

$$K = \sup_{\mathbf{e}} \mu_{\widetilde{M}}(Q,\ \mathbf{e}),$$

where $Q = PR_*$*, and* $\mu_{\widetilde{M}}(Q,\ \mathbf{e}) = \frac{\langle \widetilde{M}(I-Q)\mathbf{e},\ (I-Q)\mathbf{e}\rangle}{\langle A\mathbf{e},\ \mathbf{e}\rangle}$ *is the measure used in [12].*

Proof. One has, assuming that all inverses below exist,

$$\begin{aligned} P^T\widetilde{M} &= \left[-A_{cf}M_{ff}^{-T},\ I\right] M^T \left(M + M^T - A\right)^{-1} M \\ &= \left[-A_{cf}M_{ff}^{-T},\ I\right] \begin{bmatrix} M_{ff}^T & 0 \\ A_{cf} & M_{cc}^T \end{bmatrix} \left(D^T + D - D_A\right)^{-1} M \\ &= \left[0,\ M_{cc}^T\right] \left(D^T + D - D_A\right)^{-1} M \\ &= \left[0,\ M_{cc}^T \left(M_{cc}^T + M_{cc} - A_{cc}\right)^{-1}\right] \begin{bmatrix} M_{ff} & A_{fc} \\ 0 & M_{cc} \end{bmatrix} \\ &= \left[0,\ M_{cc}^T \left(M_{cc}^T + M_{cc} - A_{cc}\right)^{-1} M_{cc}\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} \widetilde{M}_c &= P^T\widetilde{M}P = \left[0,\ M_{cc}^T \left(M_{cc}^T + M_{cc} - A_{cc}\right)^{-1} M_{cc}\right] \begin{bmatrix} -M_{ff}^{-1}A_{fc} \\ I \end{bmatrix} \\ &= M_{cc}^T \left(M_{cc}^T + M_{cc} - A_{cc}\right)^{-1} M_{cc}. \end{aligned}$$

Finally,

$$\begin{aligned} R_* &= \widetilde{M}_c^{-1} P^T \widetilde{M} \\ &= M_{cc}^{-1}(M_{cc}^T + M_{cc} - A_{cc})M_{cc}^{-T} \left[0,\ M_{cc}^T \left(M_{cc}^T + M_{cc} - A_{cc}\right)^{-1} M_{cc}\right] \\ &= [0,\ I]. \end{aligned}$$

That is, $R = R_*$ is the simple injection mapping. The rest follows from the definition of $S_* = I - \pi_{\widetilde{M}} = I - PR_* = I - Q$ and Theorem 4.3. □

In the general case, for any well-conditioned smoother $\widetilde{M}$ (as the standard ones are, consider for example Richardson or overlapping additive Schwarz method), the entries of $\widetilde{M}_c^{-1}$ will have a fast decay rate away from the diagonal (similar to the inverse of a finite element mass-matrix). The latter result can be rigorously proved (cf., e.g., [10]). Therefore, reasonably sparse approximations R to R_* will be available, and hence using such a sparse R in practice will be a feasible and accurate enough choice. Of course, using an approximation, as indicated in

Remark 4.1, will only give upper bounds for K. More specifically, we will have $K \le \frac{1}{\lambda_{\min}}$ by finding (accurate) bounds of the minimal eigenvalue $\lambda_{\min}$ of the generalized eigenvalue problem

$$A\mathbf{q} = \lambda(I - Q^T)\widetilde{M}(I - Q)\mathbf{q}$$

where $Q = PR$. Notice that the above eigenvalue problem takes a particularly appealing form in the case of Example 4.1, but this is left for future study and is not considered further in this paper. A somewhat simpler approach for estimating K is found in Theorem 5.1.

Corollary 4.3. *The following three statements are equivalent, and they are necessary conditions for uniform convergence of the two grid method:*

1. $(I - \pi_{\widetilde{M}})$ *is bounded in A-norm*

$$\left((I - \pi_{\widetilde{M}})\mathbf{v}\right)^T A(I - \pi_{\widetilde{M}})\mathbf{v} \le K\ \mathbf{v}^T A\mathbf{v}.$$

2. $\pi_{\widetilde{M}}$ *is bounded in A-norm*

$$\left(\pi_{\widetilde{M}}\mathbf{v}\right)^T A\pi_{\widetilde{M}}\mathbf{v} \le K\ \mathbf{v}^T A\mathbf{v}.$$

3. *The spaces Range*$(S) \equiv$ *Range*$(I - \pi_{\widetilde{M}})$ *and Range*(P) *have a non-trivial angle in A-inner product, that is,*

$$\left(\mathbf{v}_s^T S^T AP\mathbf{x}\right)^2 \le (1 - \frac{1}{K})\ \mathbf{v}_s^T A_s\mathbf{v}_s\ \mathbf{x}^T A_c\mathbf{x},\ \textit{for any}\ \mathbf{v}_s\ \textit{and}\ \mathbf{x}.$$

Proof. We first give an argument that the three statements are equivalent. Consider the quadratic form $Q(t) = (\pi_{\widetilde{M}}\mathbf{v} + tS\mathbf{v}_s)^T A(\pi_{\widetilde{M}}\mathbf{v} + tS\mathbf{v}_s) - \frac{1}{K}\left(\pi_{\widetilde{M}}\mathbf{v}\right)^T A\pi_{\widetilde{M}}\mathbf{v}$. Note, that $\pi_{\widetilde{M}}S\mathbf{v}_s = 0$, hence $\pi_{\widetilde{M}}(\mathbf{v} + tS\mathbf{v}_s) = \pi_{\widetilde{M}}\mathbf{v}$. This shows that $Q(t) \ge 0$ for any real t if $\pi_{\widetilde{M}}$ is A-bounded. Then the fact that its discriminant is non-positive, shows the strengthened Cauchy-Schwarz inequality since Range$(\pi_{\widetilde{M}})$ = Range(P). The argument goes both ways. Namely, the strengthened Cauchy-Schwarz inequality implies that the discriminant is non-positive, hence Q is non-negative, that is, $\pi_{\widetilde{M}}$ is bounded in energy. Due to the symmetry of the strengthened Cauchy-Schwarz inequality one sees that $I - \pi_{\widetilde{M}}$ has the same energy norm as $\pi_{\widetilde{M}}$ (if $\pi_{\widetilde{M}} \ne I,\ 0$). That these are necessary conditions follows from Theorem 4.3. □

5. Algebraic two-grid methods and preconditioners

Corollaries 4.2 and 4.3 represent the main foundation for constructing efficient two-grid preconditioners. Namely, one needs a coarse space Range(P) such that there is a complementary one, Range(S), with the properties:

(i) the symmetrized smoother restricted to the subspace Range(S), i.e., $S^T\widetilde{M}S$, is spectrally equivalent to the subspace matrix $S^T AS$, and
(ii) the complementary spaces Range(S) and Range(P) have a non-trivial angle in A-inner product; that is, they are almost A-orthogonal.

In practice, one needs a sparse matrix P so that the coarse matrix P^TAP is also sparse, whereas explicit knowledge of the best S is not really needed. If P and S are constructed based solely on A, and similarly the smoother M (or the symmetrized one, $\widetilde{M}$) comes from a convergent splitting of A, then the resulting method (or preconditioner) belongs to the class of "algebraic" two-grid methods (or preconditioners) or simply AMG (when several coarsening levels are used). For some basic facts about AMG we refer the reader to, e.g., Ruge and Stüben [13], or the tutorial [8].

In order to guarantee the efficiency of the method, one only needs an S (not necessarily the best one defined as $\text{Range}(I-\pi_{\widetilde{M}})$) in order to test if the subspace smoother $S^T\widetilde{M}S$ is efficient on the subspace matrix S^TAS. That is, one needs an estimate (for the particular S)

$$\mathbf{v}_s^T A_s \mathbf{v}_s \le \mathbf{v}_s^T \widetilde{M}_s \mathbf{v}_s \le \kappa\, \mathbf{v}_s^T A_s \mathbf{v}_s, \tag{5.1}$$

with a reasonable constant κ. The efficiency of the smoother on the complementary space $\text{Range}(S)$ is sometimes referred to as efficient *compatible relaxation*. The latter notion is due to Achi Brandt [7].

The second main ingredient is the energy boundedness of P in the sense that for a small constant η one wants the bound,

$$\mathbf{x}^T A_c \mathbf{x} \le \eta \inf_{\mathbf{v}_s:\ \mathbf{v}=S\mathbf{v}_s+P\mathbf{x}} \mathbf{v}^T A \mathbf{v}. \tag{5.2}$$

For the simple example of $S = [\ I \ \ 0\ ]^T$ and $P = \begin{bmatrix} W \\ I \end{bmatrix}$, letting $R = [0,\ I]$, one can show (see [12]) that (5.2) is equivalent to $Q \equiv PR$ being bounded in energy, i.e.,

$$\mathbf{v}^T Q^T A Q \mathbf{v} \le \eta\, \mathbf{v}^T A \mathbf{v}.$$

Since $P: \ \ RP = I$, then $Q^2 = Q$, i.e., Q is a projection, the above estimate is equivalent (noting that $Q \ne I$ and $Q \ne 0$) to

$$((I-Q)\mathbf{v})^T A (I-Q)\mathbf{v} \le \eta\, \mathbf{v}^T A \mathbf{v}.$$

It is clear then, that a sufficient condition for P to be bounded is to establish the following "weak approximation property",

$$\|A\| \|\mathbf{v} - PR\mathbf{v}\|^2 \le \eta\, \mathbf{v}^T A \mathbf{v},$$

which was a common tool used in the classical two-(and multi-)grid convergence theory. One can actually prove the following main result in the general case (see Theorem 4.2 in [12]).

Theorem 5.1. *Assume that the estimates* (5.1) *and* (5.2) *hold true. Then the two-grid preconditioner $B = B_{TG}$ is spectrally equivalent to A with a constant $K \le \eta\kappa$.*

Proof. We have to estimate K defined in (4.9). Since $\text{Range}(S)$ is complementary to $\text{Range}(P)$ (by assumption), then any $\mathbf{v}$ can be uniquely decomposed as $\mathbf{v} = S\mathbf{v}_s + P\mathbf{x}$. The term in the numerator of (4.9) can be estimated as follows,

$$\begin{aligned} ((I-\pi_{\widetilde{M}})\mathbf{v})^T \widetilde{M} ((I-\pi_{\widetilde{M}})\mathbf{v} &= \inf_{\mathbf{y}} (\mathbf{v} - P\mathbf{y})^T \widetilde{M} (\mathbf{v} - P\mathbf{y}) \\ &\le (\mathbf{v} - P\mathbf{x})^T \widetilde{M} (\mathbf{v} - P\mathbf{x}) \\ &= \mathbf{v}_s^T S^T \widetilde{M} S \mathbf{v}_s \\ &\le \kappa\, \mathbf{v}_s^T S^T A S \mathbf{v}_s. \end{aligned}$$

In the last line above we used (5.1).

The energy boundedness in (5.2) implies a strengthened Cauchy–Schwarz inequality for Range(S) and Range(P) (see (2.11) in subsection 2.2). As also demonstrated in subsection 2.2, the Cauchy–Schwarz inequality implies the following energy boundedness of S,

$$\mathbf{v}_s^T S^T A S \mathbf{v}_s \leq \eta \inf_{\mathbf{x}:\ \mathbf{v}=S\mathbf{v}_s+P\mathbf{x}} \mathbf{v}^T A \mathbf{v}.$$

Using the projection π_A, one gets

$$\mathbf{v}_s^T S^T A S \mathbf{v}_s \leq \eta\ ((I-\pi_A)S\mathbf{v}_s)^T A (I-\pi_A) S\mathbf{v}_s.$$

Finally, since $(I-\pi_A)P\mathbf{x}=0$, one arrives at the following bound for the denominator of (4.9),

$$\begin{aligned} \mathbf{v}_s^T S^T A S \mathbf{v}_s &\leq \eta\ ((I-\pi_A)(S\mathbf{v}_s+P\mathbf{x}))^T A (I-\pi_A)(S\mathbf{v}_s+P\mathbf{x}) \\ &= \eta\ ((I-\pi_A)\mathbf{v})^T A (I-\pi_A)\mathbf{v}. \end{aligned}$$

Thus, (4.9) is finally estimated as follows

$$K = \sup_{\mathbf{v}} \frac{((I-\pi_{\widetilde{M}})\mathbf{v})^T \widetilde{M} (I-\pi_{\widetilde{M}})\mathbf{v}}{((I-\pi_A)\mathbf{v})^T A (I-\pi_A)\mathbf{v}} \leq \sup_{\mathbf{v}_s} \frac{\kappa\ \mathbf{v}_s^T S^T A S \mathbf{v}_s}{\frac{1}{\eta}\mathbf{v}_s^T S^T A S \mathbf{v}_s} = \kappa\ \eta.$$

□

6. Window based spectral AMG

In the present section we provide a purely algebraic way of selecting coarse degrees of freedom and a way to construct an energy bounded interpolation matrix P. In the analysis, we will use a simple Richardson iteration as a smoother. The presented method is an "element-free" version of the spectral AMGe method studied in Chartier et al. [9]. All definitions and constructions below are valid in the case when A is positive and only semidefinite, i.e. may have nonempty null space Null(A).

We consider the problem (2.1) and reformulate it in the following equivalent least squares minimization:

$$\mathbf{u} = \arg\min_{\mathbf{v}} \sum_w \|A_w \mathbf{v} - \mathbf{f}_w\|^2_{D_w}. \tag{6.1}$$

In the least squares formulation, each w is a subset of $\{1,\dots,n\}$, and we assume that

$$\cup w = \{1,\dots,n\},$$

where the decomposition can be overlapping. The sets w are called windows, and represent a grouping of the rows of A. The corresponding rectangular matrices we denote by A_w, i.e., $A_w = \{A_{ij}\}_{i\in w, j=1:n}$. Thus we have that $A_w \in \mathbb{R}^{|w|\times n}$, where $|\cdot|$ stands for cardinality. Accordingly in (6.1), $\mathbf{f}_w = \mathbf{f}|_w = \{\mathbf{f}_i\}_{i\in w}$ denotes a restriction of $\mathbf{f}$ to a subset and $D_w = (D_w(i))_{i\in w}$ are diagonal matrices with non–negative entries, such that for any i, $\sum_{w:\ i\in w} D_w(i) = 1$, that is, $\{D_w\}_w$ provide a partition of unity. Vanishing the first variation of the least squares functional, we obtain that the solution to the minimization problem (6.1) satisfies

$$\sum_w (A_w)^T D_w A_w \mathbf{u} = \sum_w (A_w)^T D_w \mathbf{f}_w. \tag{6.2}$$

With the specific choice of $\{D_w\}_w$, it is clear that (6.1) is equivalent to the standard least–squares problem, that is,

$$\sum_w \|A_w \mathbf{v} - \mathbf{f}_w\|^2_{D_w} = \|A\mathbf{v} - \mathbf{f}\|^2.$$

Therefore we obtain the identity,

$$\mathbf{v}^T \left(\sum_w (A_w)^T D_w A_w \right) \mathbf{v} = \mathbf{v}^T A^T A \mathbf{v}. \tag{6.3}$$

We emphasize that we will not solve the equivalent least squares problem (6.2) and it has only been introduced as a motivation to consider the "local" matrices $(A_w)^T D_w A_w$ as a tool for constructing sparse (and hence local) interpolation mapping P which we explain below. Of interest will be the Schur complements S_w, that are obtained from the matrices $(A_w)^T D_w A_w$ by eliminating the entries outside w. More specifically, let (after proper reordering of the columns of A_w),

$$A_w = \left[\begin{array}{cc} A_{ww} & A_{w,\chi} \end{array} \right] \tag{6.4}$$

where A_{ww} is the square principal submatrix of A corresponding to the subset w and $A_{w,\chi}$ corresponds to the remaining columns of A_w with indices outside w. Then, S_w is characterized by the identity

$$\mathbf{v}_w^T S_w \mathbf{v}_w = \inf_{\mathbf{v}_\chi} \left[\begin{array}{c} \mathbf{v}_w \\ \mathbf{v}_\chi \end{array} \right]^T (A_w)^T D_w A_w \left[\begin{array}{c} \mathbf{v}_w \\ \mathbf{v}_\chi \end{array} \right]. \tag{6.5}$$

An explicit expression for S_w is readily available. Let $A^T_{w,\chi} D_w A_{w,\chi} = Q^T \Lambda Q$ with $Q^T = Q^{-1}$ and $\Lambda = \text{diag}\ (\lambda)$ being diagonal matrix with eigenvalues which are non–negative. Letting $\Lambda^+ = \text{diag}\ (\lambda^+)$, where $\lambda^+ = 0$ if $\lambda = 0$, and $\lambda^+ = \lambda^{-1}$ if $\lambda > 0$, one has the expression,

$$S_w = (A_{ww})^T D_w A_{ww} - (A_{ww})^T D_w A_{w,\chi} Q^T \Lambda^+ Q A^T_{w,\ \chi} D_w A_{ww}.$$

Note that S_w is symmetric and positive semidefinite by construction (see (6.5)), and one has the inequality

$$(\mathbf{v}_w)^T S_w \mathbf{v}_w \leq \mathbf{v}^T (A_w)^T D_w A_w \mathbf{v}, \quad \mathbf{v}_w = \mathbf{v}|_w\,.$$

Hence

$$\sum_w (\mathbf{v}_w)^T S_w \mathbf{v}_w \leq \mathbf{v}^T \left(\sum_w (A_w)^T D_w A_w \right) \mathbf{v} = \mathbf{v}^T A^T A \mathbf{v} \leq \|A\|\ \mathbf{v}^T A \mathbf{v}. \tag{6.6}$$

This inequality implies (letting $\mathbf{v} = 0$ outside w) that $(\mathbf{v}_w)^T S_w \mathbf{v}_w \leq \|A\|\ \mathbf{v}^T A \mathbf{v} \leq \|A\|^2\ \mathbf{v}^T \mathbf{v} = \|A\|^2\ \mathbf{v}_w^T \mathbf{v}_w$, that is,

$$\|S_w\| \leq \|A\|^2. \tag{6.7}$$

Selecting coarse degrees of freedom

Our goal will be to select a coarse space. The way we do that will be by fixing a window and associating with it a number $m_w \leq |w|$. Then we construct m_w basis vectors (columns of P) corresponding to this window in the following way: All the eigenvectors and eigenvalues of S_w are computed and the eigenvectors corresponding to the first m_w eigenvalues are chosen. Since generally the windows have overlap, another partition of unity is constructed, with non–negative diagonal matrices $\{Q_w\}$ where each Q_w is non-zero only on w and the set $\{Q_w\}$

satisfies $\sum_w Q_w = I$. From the first m_w eigenvectors of S_w extended by zero outside w, we form column-wise the local interpolation matrix P_w which hence has m_w columns. The global interpolation matrix is then defined as

$$P = \sum_w Q_w \, [0, \ P_w, \ 0] \, .$$

Here, for a global coarse vector $\mathbf{v}^c = (\mathbf{v}^c_w)$, the action of $[0, \ P_w, \ 0]$ is defined such that $[0, \ P_w, \ 0] \, \mathbf{v}^c = P_w(\mathbf{v}^c|_w) = P_w \mathbf{v}^c_w$.

The remainder of this section follows the presentation in [9], but the main result (Theorem 6.1) utilizes our main identity (4.9). The first result concerns the null-space of A, namely, that it is contained in the range of the interpolation P.

Lemma 6.1. *Suppose that m_w is such that $m_w \geq$ dim Null (S_w) for every window w. Then Null $(A) \subset$ Range (P), that is, if $A\mathbf{v} = 0$, then there exists a $\mathbf{v}^c \in \mathbb{R}^{n_c}$ such that $\mathbf{v} = P\mathbf{v}^c$.*

Proof. Let $A\mathbf{v} = 0$. Then from inequality (6.6) it follows that $S_w \mathbf{v}_w = 0$, where $\mathbf{v}_w = \mathbf{v}|_w$ and we extend $\mathbf{v}_w$ by zero outside w whenever needed. Hence, by our assumption on m_w there exists a local coarse grid vector $\mathbf{v}^c_w$ such that $\mathbf{v}_w = P_w \mathbf{v}^c_w$. Let $\mathbf{v}^c$ be the composite coarse grid vector that agrees with $\mathbf{v}^c_w$ on w, for each w. This is simply the collection $\mathbf{v}^c = (\mathbf{v}^c_w)$. Then,

$$P\mathbf{v}^c = \sum_w Q_w P_w \mathbf{v}^c_w = \sum_w Q_w \mathbf{v}_w = \sum_w Q_w \mathbf{v} = \mathbf{v}.$$

$\square$

Two-grid convergence

First, we prove a main coarse-grid "weak approximation property".

Lemma 6.2. *Assume that the windows $\{w\}$ are selected in a "quasi-uniform" manner such that for all w the following uniform estimate holds:*

$$\|S_w\| \geq \eta \|A\|^2. \tag{6.8}$$

Note that $\eta \leq 1$ (see (6.7)). Assume that we have chosen m_w so well that for a constant $\delta > 0$ uniformly in w one has

$$\|S_w\| \leq \delta \lambda_{m_w+1}(S_w), \tag{6.9}$$

where $\lambda_{m_w+1}(S_w)$ denotes the (m_w+1)th smallest eigenvalue of S_w. It is clear that $\delta \geq 1$. Then, for any vector $\mathbf{e} \in \mathbb{R}^n$, there exists a global interpolant ϵ in the range of P such that

$$(\mathbf{e} - \epsilon)^T A(\mathbf{e} - \epsilon) \leq \|A\| \|\mathbf{e} - \epsilon\|^2 \leq \frac{\delta}{\eta} \, \mathbf{e}^T A\mathbf{e}. \tag{6.10}$$

Before we present the proof of the lemma, we would like to illustrate how the assumptions (6.8) and (6.9) can be verified. Consider the simple example, when A corresponds to a finite element discretization of the Laplace operator on uniform triangular mesh on the unit square domain Ω with Neumann boundary conditions. One first notices that the entries of A are mesh independent. Therefore $\|A\|$ is bounded above by a mesh–independent constant ($\|A\| \leq 8$). Let $h = \frac{1}{m_0 m}$ be the fine–grid mesh size for a given integer m and a fixed (independently of

m) integer $m_0 > 1$. Let $H = \frac{1}{m}$. This implies that Ω can be covered exactly by m^2 equal coarse rectangles of size $H = m_0 h$. Each coarse rectangle defines a window as the set of indices corresponding to the fine–grid nodes contained in that coarse rectangle. There are $(m_0+1)^2$ nodes per rectangle and all the rectangles form an overlapping partition of the grid. A simple observation is that any such rectangle can have 0, 1 or 2 common sides with the boundary of Ω and therefore, there are only three different types of window matrices A_w and respective Schur complements S_w. It is clear then that inequalities of the type (6.8) and (6.9) are feasible for a mesh independent constant η and for a mesh–independent choice of m_w. For the simple example in consideration, fix $m_0 > 3$, hence $(m_0+1)^2 > 4(m_0+1)$; that is, let the number of nodes in w be larger than the number of its outside boundary nodes (i.e., nodes outside w, that are connected to w through non–zero entries of $A_{w,\,\chi}$). From (6.5) it is clear that if $S_w \mathbf{v}_w = 0$ then there is a $\mathbf{v}_\chi$ such that $A_{ww}\mathbf{v}_w + A_{w,\chi}\mathbf{v}_\chi = 0$. Since in our case A_{ww} is invertible, one has then that the dimension of the null-space of S_w equals the dimension of Range$(A_{w,\,\chi})$. The latter is bounded above by $4(m_0+1)$ (which is the number of nodes outside w that are connected to w through non–zero entries of $A_{w,\,\chi}$). Therefore, we may choose any fixed integer $m_w \geq 4(m_0+1)$ (and $m_w < (m_0+1)^2$) to guarantee estimate (6.9) since then $\lambda_{m_w+1}(S_w) > 0$. The number of coarse degrees of freedom (or dofs) then equals $m_w m^2$. This implies that the coarsening factor, defined below, will satisfy

$$\frac{\#\text{ fine dofs}}{\#\text{ coarse dofs}} = \frac{(mm_0+1)^2}{m_w m^2} = \frac{(m_0+\frac{1}{m})^2}{m_w} \simeq \frac{m_0^2}{m_w}.$$

For example, if we choose $m_w = 4(m_0+1)$, the coarsening factor is $\simeq \frac{m_0}{4+\frac{4}{m_0}}$. It is strictly greater than one if $m_0 > 4$, and it can be made as large as needed by increasing m_0. (The latter, of course, reflects the size of the windows.) In conclusion, in this simple example, one can easily see that the bounds $\eta = \min_w \frac{\|S_w\|}{\|A\|^2} \leq 1$ and $\delta = \max_w \frac{\|S_w\|}{\lambda_{m_w+1}(S_w)} \geq 1$ are fixed mesh independent constants. This is true, since the matrices S_w are finite number, the number m_w is fixed, and therefore the eigenvalues $\lambda_{m_w+1}(S_w)$ are also finite number, and all these numbers have nothing to do with m (or the mesh size $h \mapsto 0$). Similar reasoning can be applied to more general quasi-uniform meshes. This is the case if the windows can be chosen such that the matrices $(A_w)^T D_w A_w$ and S_w are spectrally equivalent to a finite number of mesh–independent reference ones. The constants in the spectral equivalence then will only depend on the angles in the mesh.

Proof of Lemma 6.2. The analysis follows [9]. Let $\mathbf{e} \in \mathbb{R}^n$ be given. Note that our assumption on m_w is equivalent to the assumption that, for any window w, there exists a ϵ_w in the range of P_w such that

$$\|S_w\|\|\mathbf{e}_w - \epsilon_w\|^2 \leq \delta\ \mathbf{e}_w^T S_w \mathbf{e}_w, \qquad (6.11)$$

where $\mathbf{e}_w = \mathbf{e}|_w$ and whenever needed we consider $\mathbf{e}_w$ and ϵ_w extended by zero outside w. We now construct an ϵ in the range of P which will satisfy (6.10). Namely, we set $\epsilon = \sum_w Q_w \epsilon_w$.

One notices that $\sum_w Q_w \epsilon = \epsilon = \sum_w Q_w \epsilon_w$. Hence,

$$\begin{aligned}
\|\mathbf{e}-\epsilon\|^2 &= (\mathbf{e}-\epsilon)^T \left(\sum_w Q_w (\mathbf{e}-\epsilon) \right) \\
&= (\mathbf{e}-\epsilon)^T \left(\sum_w Q_w (\mathbf{e}_w-\epsilon_w) \right) \\
&= \sum_w \left(Q_w^{\frac{1}{2}} (\mathbf{e}-\epsilon) \right)^T \left(Q_w^{\frac{1}{2}} (\mathbf{e}_w-\epsilon_w) \right) \\
&\le \left[\sum_w (\mathbf{e}-\epsilon)^T Q_w (\mathbf{e}-\epsilon) \right]^{\frac{1}{2}} \left[\sum_w \|Q_w^{\frac{1}{2}} (\mathbf{e}_w-\epsilon_w)\|^2 \right]^{\frac{1}{2}} \\
&= \|\mathbf{e}-\epsilon\| \left[\sum_w \|Q_w^{\frac{1}{2}} (\mathbf{e}_w-\epsilon_w)\|^2 \right]^{\frac{1}{2}}.
\end{aligned}$$

That is,

$$\|\mathbf{e}-\epsilon\|^2 \le \sum_w \|Q_w^{\frac{1}{2}} (\mathbf{e}_w-\epsilon_w)\|^2.$$

Therefore, based on (6.11), the quasi-uniformity of $\{w\}$, and inequality (6.6), one gets

$$\begin{aligned}
\|\mathbf{e}-\epsilon\|^2 &\le \sum_w \|Q_w^{\frac{1}{2}} (\mathbf{e}_w-\epsilon_w)\|^2 \le \sum_w \|\mathbf{e}_w-\epsilon_w\|^2 \\
&\le \delta \sum_w \frac{\mathbf{e}_w^T S_w \mathbf{e}_w}{\|S_w\|} \le \frac{\delta}{\eta\|A\|^2} \sum_w \mathbf{e}_w^T S_w \mathbf{e}_w \\
&\le \frac{\delta}{\eta\|A\|^2} \mathbf{e}^T A^T A \mathbf{e} \le \frac{\delta}{\eta\|A\|} \mathbf{e}^T A \mathbf{e}.
\end{aligned}$$

□

We will use estimate (6.10) to show that the two-grid method with the Richardson iteration matrix $M = \frac{\|A\|}{\omega} I$, $\omega \in (0,2)$, which leads to $\widetilde{M} = M(2M-A)^{-1}M = \frac{\|A\|^2}{\omega^2}(2\frac{\|A\|}{\omega} I - A)^{-1}$, is uniformly convergent. More specifically, we have the following main spectral equivalence result.

Theorem 6.1. *The algebraic two-grid preconditioner B, based on the Richardson smoother $M = \frac{\|A\|}{\omega} I$, $\omega \in (0,2)$, and the coarse space based on P constructed by the window spectral AMG method, is spectrally equivalent to A and the following estimate holds:*

$$\mathbf{v}^T A \mathbf{v} \le \mathbf{v}^T B \mathbf{v} \le \frac{\delta}{\eta\omega(2-\omega)} \mathbf{v}^T A \mathbf{v}.$$

The term $\frac{\delta}{\eta}$ comes from the coarse-grid approximation property (6.10).

Proof. One first notices that

$$\mathbf{w}^T \widetilde{M} \mathbf{w} = \frac{\|A\|^2}{\omega^2} \mathbf{w}^T \left(2\frac{\|A\|}{\omega} I - A \right)^{-1} \mathbf{w} \le \frac{\|A\|}{\omega(2-\omega)} \mathbf{w}^T \mathbf{w} = \frac{1}{2-\omega} \mathbf{w}^T M \mathbf{w}.$$

Then, based on the $\widetilde{M}$-norm minimization property of the projection $\pi_{\widetilde{M}}$, one has,

$$\begin{aligned}((I-\pi_{\widetilde{M}})\mathbf{v})^T\widetilde{M}(I-\pi_{\widetilde{M}})\mathbf{v} &= \inf_{\epsilon\in\text{Range}(P)} (\mathbf{v}-\epsilon)^T\widetilde{M}(\mathbf{v}-\epsilon)\\ &\leq \frac{1}{2-\omega}\inf_{\epsilon\in\text{Range}(P)} (\mathbf{v}-\epsilon)^T M(\mathbf{v}-\epsilon)\\ &= \frac{\|A\|}{\omega(2-\omega)}\inf_{\epsilon\in\text{Range}(P)} \|\mathbf{v}-\epsilon\|^2\\ &\leq \frac{1}{\omega(2-\omega)}\;\frac{\delta}{\eta}\;\mathbf{v}^TA\mathbf{v}.\end{aligned}$$

Thus based on Theorem 4.3 we have that the corresponding two-grid preconditioner B is spectrally equivalent to A with a constant

$$K = \sup_{\mathbf{v}} \frac{((I-\pi_{\widetilde{M}})\mathbf{v})^T\widetilde{M}(I-\pi_{\widetilde{M}})\mathbf{v}}{\mathbf{v}^TA\mathbf{v}} \leq \frac{\delta}{\eta\,\omega(2-\omega)}.$$

□

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